

Coupling Independence Implies Zero-Freeness

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Abstract

For $\Delta \geq 2$ and $q \geq 11\Delta/6$, we prove that the antiferromagnetic q -state Potts partition function on finite simple graphs of maximum degree at most Δ has no Fisher zeros in a graph-uniform complex neighborhood of $[0, 1]$. Above the threshold, the proof establishes coupling independence throughout $[0, 1]$ using a soft version of Vigoda’s flip dynamics. A separator-shell argument then converts this probabilistic control into complex zero-freeness. We also obtain Lee–Yang zero-freeness for vertex- and edge-colour fields and a zero-free neighborhood of the nonnegative orthant for log-concave Boolean Holant problems whose signatures satisfy $f(0) > 0$, including b -matchings. Seven further Potts regimes obtained from the same framework are summarized in the appendix.

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1 Introduction

For a finite simple graph $G = (V, E)$ and a positive integer q , write $[q] = \{1, \dots, q\}$ and define

$$Z_G(q; z) := \sum_{\sigma: V \rightarrow [q]} z^{m_G(\sigma)}, \quad m_G(\sigma) := |\{uv \in E : \sigma_u = \sigma_v\}|. \quad (1)$$

The real interval $[0, 1]$ describes the antiferromagnetic model: $Z_G(q; 0)$ counts proper q -colourings, while $Z_G(q; 1) = q^{|V|}$. The complex zeros of this edge-activity polynomial are often called *Fisher zeros*. The study of partition-function zeros originates in the work of Lee and Yang and of Fisher on phase transitions [LY52, Fis65]; an algorithmic motivation is that a zero-free neighborhood of $[0, 1]$, uniform over graphs of bounded degree, permits deterministic approximation by interpolation [Bar16, PR17].

1.1 Historical context and related work

Sampling and deterministic counting. Approximately counting proper graph colourings is a central problem in algorithmic combinatorics. The sampling approach began with Jerrum’s rapid-mixing bound for Glauber dynamics when $q > 2\Delta$ [Jer95]. Vigoda introduced bounded two-colour flips and reduced the threshold to $q > 11\Delta/6$ [Vig00]. After nearly two decades, Chen, Delcourt, Moitra, Perarnau, and Postle improved the coefficient to $11/6 - \varepsilon_0$, with $\varepsilon_0 \approx 10^{-5}$ [CDM⁺19]; Carlson and Vigoda subsequently reached $q \geq 1.809\Delta$ for $\Delta \geq 125$ [CV25]. Sampling yields randomized approximate counting through self-reduction [JVV86].

Early deterministic progress came from *correlation decay*, which approximates conditional marginals by truncating recursive computations [BG08, Wei06]. Gamarnik and Katz obtained an FPTAS for triangle-free graphs at a colour-to-degree ratio of about 2.8432 [GK12]; Lu and Yin reduced the ratio to about 2.5807 on general graphs [LY13]. Barvinok’s *interpolation* method instead approximates $\log Z$ in a zero-free region, with polynomial-time coefficient computation on bounded-degree graphs supplied by Patel and Regts [Bar16, PR17].

Fisher zero-freeness. Bencs, Davies, Patel, and Regts obtained a zero-free neighborhood of $[0, 1]$ for $q \geq e\Delta + 1$, allowing independent edge activities [BDPR21, Theorem 1.1 and Remark 1.1]. Liu, Sinclair, and Srivastava brought the univariate threshold to $q \geq 2\Delta$ [LSS25]. Their method extends real recurrence estimates to complex activities; using the spatial-mixing estimates of Gamarnik, Katz, and Misra [GKM15], it also gives the stronger triangle-free bound in Table 1. Bencs, Berrekkal, and Regts subsequently gave a more transparent log-ratio proof and exploited the local structure around a root to cross the 2Δ barrier, obtaining $q \geq 1.998\Delta$ [BBR26]. Their result first appeared in 2024. These are bounds in the edge-activity variable; chromatic-root bounds in the complex q -plane address a complementary problem [Sok01, FP08, JPR24, BR25].

Coupling independence and counting. Following the introduction of spectral independence [ALOG20], coupling bounds were used to control influences in several spin systems [FGYZ22, Liu21, BCC⁺22]. As Chen, Feng, Guo, Zhang, and Zou point out, these earlier arguments already implicitly establish the stronger coupling-independence property [CFG⁺25, Section 1.1]. Chen and Zhang then formalized coupling independence for Boolean systems [CZ23]; Chen, Feng, Guo, Zhang, and Zou extended the definition to finite-spin systems and made it an explicit input to deterministic approximate counting [CFG⁺25]. Their algorithm combines recursive marginal approximation

with coupling-based linear programs, building on the approach introduced by Moitra [Moi19]. For permissive bounded-degree systems, coupling independence and a lower bound on nonzero conditional marginals suffice for an FPTAS [CFG⁺25, Theorem 4]. Applied to the contractive flip couplings of Chen, Delcourt, Moitra, Perarnau, and Postle and of Carlson and Vigoda [CDM⁺19, CV25], their theorem gives FPTASes at $q \geq (11/6 - \varepsilon_0)\Delta$ and at $q \geq 1.809\Delta$ for $\Delta \geq 125$, closing the gap with the general-graph sampling bounds [CFG⁺25, Corollary 6].

Zero-freeness, correlation decay, and rapid mixing. Connections between spatial mixing and analyticity go back to Dobrushin and Shlosman [DS85, DS87]. Subsequent work establishes implications between correlation decay and zero-freeness for various bounded-degree systems [LSS19, LSS25, SS21, PR19, PR20, Gam23, Reg23, SY24, CSS26]. Shao and Shi further derive Weitz-type approximation and spatial-mixing consequences from zero-freeness [SS26]. Zero-freeness can also imply rapid mixing via spectral independence [ALOG20, CLV24], while recent work derives zero-freeness by complexifying rapidly mixing Markov-chain arguments [LWYY25, LY26].

1.2 Main results

Our main theorem gives a uniform zero-free neighborhood of the entire Potts interval $[0, 1]$ when $q \geq 11\Delta/6$. Table 1 compares this threshold with earlier results. We state the result for pinned partition functions, since conditioning is essential to the coupling argument. Let $\tau: \Lambda \rightarrow [q]$ be a precolouring, and let $m_G(\tau)$ count monochromatic edges of $G[\Lambda]$. Define

$$\hat{Z}_G^\tau(q; z) := \sum_{\substack{\sigma: V \rightarrow [q] \\ \sigma|_\Lambda = \tau}} z^{m_G(\sigma) - m_G(\tau)}. \quad (2)$$

This polynomial removes the contribution of pinned-only edges from the usual pinned partition function:

$$Z_G^\tau(q; z) = z^{m_G(\tau)} \hat{Z}_G^\tau(q; z). \quad (3)$$

It is defined even at $z = 0$ and for improper pinnings; for the empty pinning it equals $Z_G(q; z)$. We suppress q when fixed and write $\mathcal{U}_\varepsilon(I) = \{z \in \mathbb{C} : \text{dist}(z, I) < \varepsilon\}$.

Theorem 1.1 (Potts zero-freeness near $[0, 1]$). *Let q, Δ be integers with $\Delta \geq 2$ and $q \geq 11\Delta/6$. There exists $\varepsilon = \varepsilon(q, \Delta) > 0$ such that, for every finite simple graph G of maximum degree at most Δ and every pinning τ ,*

$$\hat{Z}_G^\tau(z) \neq 0 \quad \text{for every } z \in \mathcal{U}_\varepsilon([0, 1]).$$

Consequently, $Z_G^\tau(z)$ has no zeros in the same neighborhood except possibly at $z = 0$, where its zero has multiplicity $m_G(\tau)$. This applies in particular to Z_G and to Z_G^τ whenever τ is proper on $G[\Lambda]$.

Work	Graph class	Parameters
Bencs, Davies, Patel, and Regts [BDPR21]	general	$q \geq e\Delta + 1$
Liu, Sinclair, and Srivastava [LSS25]	general	$\Delta \geq 3, q \geq 2\Delta$
Liu, Sinclair, and Srivastava [LSS25]	triangle-free	$\Delta \geq 3, q \geq \alpha\Delta + \beta(\alpha), \alpha > \alpha_\star \approx 1.7633$
Bencs, Berrekkal, and Regts [BBR26]	general	$q, \Delta \geq 3, q \geq 1.998\Delta$
This paper	general	$\Delta \geq 2, q \geq 11\Delta/6$

Here $\alpha_\star \approx 1.7633$ solves $\alpha_\star = e^{1/\alpha_\star}$, and $\beta(\alpha)$ is the additive constant in the triangle-free result of [LSS25, Theorem 1.3]. All rows imply the stated univariate zero-freeness; the first work also allows independent complex edge activities.

Table 1: Selected uniform Fisher zero-free neighborhoods of $[0, 1]$ for the Potts model.

The theorem follows from a transfer principle whose main probabilistic hypothesis is *Hamming coupling independence*: under every further pinning, two conditional Gibbs laws that differ in the colour fixed at one site admit a coupling with bounded expected Hamming disagreement on the remaining sites, independently of the size of the graph. For any class of bounded-degree graphs closed under induced subgraphs, with $q \geq \Delta + 1$, Theorem 2.4 converts this property at $x = 0$, together with a uniform bound on each $[\delta, 1]$, $\delta > 0$, into zero-freeness near $[0, 1]$. In general, unlike the coupling-independence-to-counting framework, our coupling-independence-to-zero-freeness transfer does not require a uniform lower bound on nonzero conditional marginals. We establish the coupling hypothesis using a soft version of Vigoda’s flip dynamics. For $q > 11\Delta/6$ the coupling bound is uniform on all of $[0, 1]$; at equality we combine the positive-temperature bound with the known hard-colouring bound [CFG⁺25, Theorem 20].

For fixed (q, Δ) and rational $x \in [0, 1]$, interpolation from $Z_G(q; 1) = q^{|V|}$, with the coefficient algorithm of Patel and Regts, gives a fully polynomial-time approximation scheme (FPTAS) for $Z_G(q; x)$ [Bar16, PR17]. Its running time is polynomial in $|V|$ and the inverse relative accuracy. The radius can be chosen effectively, since the bounded local systems used in the proof form a finite, effectively enumerable family. At $x = 0$, stronger counting thresholds are already available from [CFG⁺25]; the new conclusion is the uniform exclusion of complex zeros throughout the Potts interval.

The same shell argument applies to other local models once the relevant pinning and local-weight conditions are verified. For proper vertex and edge colourings, it gives Lee–Yang zero-freeness under independent colour-field perturbations near the all-one vector. The vertex-colouring result uses coupling bounds at $q \geq (11/6 - 1/84000)\Delta$, at $q \geq 1.809\Delta$ for $\Delta \geq 125$, and at $q \geq \Delta + 3$ for sufficiently large girth [CFG⁺25, CLMM23]; the edge-colouring result uses $q \geq 3\Delta$ [CWZZ25]. We obtain these inputs at $x = 0$ from the appendix coupling theorems, so the field results share their formalized proofs and literature dependencies. For log-concave Boolean Holant problems with $f(0) > 0$, we turn the couplings of Chen and Gu [CG24] into zero-free neighborhoods of the nonnegative activity orthant. Deterministic counting for these Holant problems was already established by He, Li, Qiu, and Zhang [HLQZ25]. Table 2 summarizes these conclusions; precise statements appear in Section 4. Further Potts regimes are recorded in Section A.

Model	Parameters	Zero-free near	Result
AF Potts	$\Delta \geq 2, q \geq 11\Delta/6$	$[0, 1]$	Thm. 1.1
vertex-colour fields	the three colouring regimes above	$\lambda = \mathbf{1}$	Thm. 4.1
edge-colour fields	$\Delta \geq 2, q \geq 3\Delta$	$\lambda = \mathbf{1}$	Cor. 4.2
log-concave Boolean Holant	$\Delta \geq 2, \mathcal{F}$ finite, $f(0) > 0$ for all $f \in \mathcal{F}$	$[0, \infty)^E$	Thm. 4.4
b -matchings	$\Delta \geq 2$	$[0, \infty)^E$	Cor. 4.5
b -edge covers	$\Delta \geq 2$	$(0, \infty)^E$	Cor. 4.6

For the last three rows, the width is uniform over graphs on each fixed compact activity box; taking the union of these neighborhoods gives the stated orthant. For fields, $\mathbf{1}$ is the all-one vector and the polydisc radius depends only on (q, Δ) .

Table 2: Zero-free neighborhoods proved in the main text.

1.3 Proof overview

The first step is the averaging argument used by Chen et al. to prove total-influence decay [CFG⁺25, Theorem 12]. If a coupling has expected total Hamming disagreement at most C , then among

L disjoint distance shells around a root there is one with expected disagreement at most C/L . Choosing L large enough gives a separator with a bounded interior and a small coupling cost on its boundary.

The complex step is to write each conditional partition-function response as an average under the corresponding *real* shell law. Induction makes the logarithmic response of the common exterior Lipschitz in the shell colouring, and the bounded interior contributes a uniformly small perturbation. Our complex-average stability lemma converts the small Hamming coupling cost into control of the ratio of these averages. This closes the induction proving nonvanishing, in the style of earlier Potts zero-free proofs [LSS25, BBR26]. At $x = 0$, previously forbidden configurations acquire nonzero weight under a complex perturbation; we bound their contribution as an additive error. This yields a disk at zero that can be joined to the neighborhoods of positive activities.

To supply the coupling input, we write each soft Potts interaction as a mixture of an unconstrained edge and a hard inequality. Conditional on the random constraints, the system is a hard list-colouring model. We adapt Vigoda's bounded-flip coupling [Vig00] to jointly sampled constraint sets and control the effect of changing a pinned boundary. Averaging the resulting contraction estimate and applying stationary comparison [CFG⁺25, Lemma 19] give the soft coupling bound. The new ingredients are this positive-temperature extension and the complex stability argument that turns coupling independence into nonvanishing.

Organization. Section 2 gives the notation and states the Potts transfer theorem. Section 3 develops the complex-average lemma, establishes the transfer and main theorems, and then proves the soft coupling bound. Section 4 establishes zero-freeness for colour fields and Holant problems; the appendix records further Potts regimes obtained from the same framework.

2 Preliminaries

2.1 Pinnings and partition functions

Throughout, q and Δ are positive integers, $[q] = \{1, \dots, q\}$, $x \in [0, 1]$ is a real base point, and $z \in \mathbb{C}$ is the complex edge-activity variable. We write $\mathbf{1}_{[A]}$ for the indicator of an event A , and use the convention $0^0 = 1$. For a finite graph G , write

$$\Delta(G) := \max_{v \in V(G)} \deg_G(v)$$

for its maximum degree; the symbol Δ denotes a fixed degree bound. Let

$$\mathcal{G}_\Delta := \{G : G \text{ is finite, simple, and } \Delta(G) \leq \Delta\},$$

and, for $g \geq 3$, let

$$\mathcal{G}_{\Delta, g} := \{G \in \mathcal{G}_\Delta : \text{girth}(G) \geq g\}.$$

Both classes are closed under induced subgraphs. We use the convention that an acyclic graph, including the empty graph, has $\text{girth} + \infty$.

Fix $G = (V, E) \in \mathcal{G}_\Delta$. For a pinning $\tau : \Lambda \rightarrow [q]$, write

$$V^\tau := V \setminus \Lambda, \quad G^\tau := G[V^\tau],$$

and call G^τ the *free graph*. For $u \in V^\tau$ and $c \in [q]$, define the boundary count

$$b_u^\tau(c) := |\{p \in \Lambda : up \in E, \tau_p = c\}|. \tag{4}$$

Boundary counts record pinned neighbours separately, even when several have the same colour. The normalized pinned partition function from (2) has the factorized form

$$\widehat{Z}_G^\tau(z) = \sum_{\varphi: V^\tau \rightarrow [q]} \prod_{\substack{uv \in E \\ \{u,v\} \not\subseteq \Lambda}} (z \mathbf{1}_{[(\tau \cup \varphi)_u = (\tau \cup \varphi)_v]} + \mathbf{1}_{[(\tau \cup \varphi)_u \neq (\tau \cup \varphi)_v]}). \quad (5)$$

Equivalently,

$$\widehat{Z}_G^\tau(z) = \sum_{\varphi: V^\tau \rightarrow [q]} \prod_{u \in V^\tau} z^{b_u^\tau(\varphi_u)} \prod_{uv \in E(G^\tau)} (z \mathbf{1}_{[\varphi_u = \varphi_v]} + \mathbf{1}_{[\varphi_u \neq \varphi_v]}).$$

The full and normalized polynomials are related by (3); in particular, no division by z is involved.

Lemma 2.1 (Pinned-leaf realization). *For every $G \in \mathcal{G}_\Delta$ and pinning τ , there is a graph $\tilde{G} \in \mathcal{G}_\Delta$ with a pinning $\tilde{\tau}$ whose domain consists of leaves, such that*

$$\widehat{Z}_G^\tau(z) = \widehat{Z}_{\tilde{G}}^{\tilde{\tau}}(z), \quad \text{girth}(\tilde{G}) = \text{girth}(G^\tau).$$

Proof. Start from G^τ . For every edge $up \in E$ with $u \in V^\tau$ and $p \in \Lambda$, attach a distinct leaf ℓ_{up} to u and pin it to τ_p . Each leaf reproduces the corresponding boundary factor in (5), so the normalized polynomials agree. The degree of each free vertex is unchanged from its degree in G , and adding leaves creates no cycle. $\square$

Whenever $\widehat{Z}_G^\tau(x) > 0$, let $\mu_{G,x}^\tau$ be the Gibbs law on $[q]^{V^\tau}$:

$$\mu_{G,x}^\tau(\varphi) := \frac{1}{\widehat{Z}_G^\tau(x)} \prod_{u \in V^\tau} x^{b_u^\tau(\varphi_u)} \prod_{uv \in E(G^\tau)} (x \mathbf{1}_{[\varphi_u = \varphi_v]} + \mathbf{1}_{[\varphi_u \neq \varphi_v]}).$$

For $x > 0$, this is the usual conditional Gibbs law after pinned-only edge contributions have been removed. At $x = 0$, it is the normalized hard pinned law, which remains meaningful even when τ is improper.

For $r \in V^\tau$ and $a \in [q]$, write

$$\tau^{r=a} := \tau \cup \{r \mapsto a\}.$$

Then $V^{\tau^{r=a}} = V^\tau \setminus \{r\}$, and

$$\widehat{Z}_G^\tau(z) = \sum_{a \in [q]} z^{b_r^\tau(a)} \widehat{Z}_G^{\tau^{r=a}}(z). \quad (6)$$

The multiplier $z^{b_r^\tau(a)}$ is determined when r is pinned. The normalized function $\widehat{Z}_G^{\tau^{r=a}}$ remains well defined even when this multiplier vanishes at $z = 0$. No additional notation is introduced for one-vertex restrictions.

2.2 Hard-colouring feasibility

A colour c is blocked at a free vertex u when $b_u^\tau(c) > 0$. Its hard effective list is

$$L^\tau(u) := \{c \in [q] : b_u^\tau(c) = 0\}.$$

A colouring $\varphi: V^\tau \rightarrow [q]$ is *hard-feasible* if it is proper on G^τ and $\varphi_u \in L^\tau(u)$ for every $u \in V^\tau$.

Lemma 2.2 (Hard feasibility). *Assume $q \geq \Delta + 1$. For every $G \in \mathcal{G}_\Delta$, every pinning τ , and every $u \in V^\tau$,*

$$|L^\tau(u)| \geq \deg_{G^\tau}(u) + 1.$$

Consequently $\widehat{Z}_G^\tau(0) > 0$.

Proof. At most $|N_G(u) \cap \Lambda|$ colours are blocked at u . Since

$$\deg_{G^\tau}(u) + |N_G(u) \cap \Lambda| = \deg_G(u) \leq \Delta,$$

we have

$$|L^\tau(u)| \geq q - |N_G(u) \cap \Lambda| \geq \deg_{G^\tau}(u) + 1.$$

A greedy list-colouring of G^τ , in any vertex ordering, leaves an available colour at every step, so a hard-feasible colouring exists and $\hat{Z}_G^\tau(0) > 0$. $\square$

2.3 Coupling independence and Wasserstein distance

We fix the Wasserstein notation and state the coupling-independence input used below. The Markov-chain comparison that proves this input is part of Section 3.

Let $(\mathcal{X}, d)$ be a finite metric space. For probability measures ν_0, ν_1 on $\mathcal{X}$,

$$W_{1,d}(\nu_0, \nu_1) = \inf_{\pi \in \Pi(\nu_0, \nu_1)} \mathbb{E}_\pi[d(X, Y)], \quad (7)$$

where $\Pi(\nu_0, \nu_1)$ denotes the set of couplings of ν_0 and ν_1 , and X, Y are the two coordinate projections under π . On a finite space the infimum is attained. For configurations on a common site set,

$$\text{Ham}(\sigma, \sigma') := |\{u : \sigma_u \neq \sigma'_u\}|, \quad W_{1,\text{Ham}} := W_{1,d} \text{ with } d = \text{Ham}.$$

All Hamming distances in the Potts sections below are taken on the common free vertex set. The Potts input used in Section 3 is the following.

Definition 2.3 (Potts coupling independence). Let $C \in [0, \infty)$, $\mathcal{G} \subseteq \mathcal{G}_\Delta$, and $J \subseteq [0, 1]$. We say that $\mathcal{G}$ satisfies *C-coupling independence* on J if, for every $G \in \mathcal{G}$, every pinning τ , every $r \in V^\tau$, all $a, b \in [q]$, and every $x \in J$, the two pinned laws are well defined and

$$W_{1,\text{Ham}}(\mu_{G,x}^{\tau^{r=a}}, \mu_{G,x}^{\tau^{r=b}}) \leq C. \quad (8)$$

They live on the common free set $V^\tau \setminus \{r\}$. At $x = 0$ they are the normalized hard pinned laws. When $\mathcal{G} = \mathcal{G}_\Delta$, we simply say that the bounded-degree Potts model satisfies *C-coupling independence*.

2.4 The transfer principle

The following theorem converts the real coupling bounds into complex zero-freeness. Its proof is given in Section 3.2. Both $\mathcal{G}_\Delta$ and $\mathcal{G}_{\Delta,g}$ are closed under taking induced subgraphs.

Theorem 2.4 (Potts transfer). *Fix integers q, Δ with $\Delta \geq 2$ and $q \geq \Delta + 1$, and let $\mathcal{G} \subseteq \mathcal{G}_\Delta$ be closed under taking induced subgraphs. Assume:*

- (i) $\mathcal{G}$ satisfies coupling independence at $x = 0$ with constant C_0 ;
- (ii) for every $\delta \in (0, 1]$, it satisfies coupling independence on $[\delta, 1]$ with a finite constant C_δ .

Then there is $\varepsilon > 0$ such that, for every $G \in \mathcal{G}$ and every pinning τ ,

$$\hat{Z}_G^\tau(z) \neq 0 \quad \text{whenever } \text{dist}(z, [0, 1]) < \varepsilon.$$

Consequently, $Z_G^\tau(z) = z^{m_G(\tau)} \hat{Z}_G^\tau(z)$ has only the additional forced zero at $z = 0$ when $m_G(\tau) \geq 1$.

Beyond Potts. The theorem is written in Potts notation to keep the statement concrete. The shell comparison uses a finite local state space, bounded local interactions, and the Hamming coupling bound. Extending the transfer to another model also requires closure under the relevant pinnings and uniform control of the local factors, including those that vanish at a real base point. We verify these properties for the applications below.

3 Potts zero-freeness for $q \geq 11\Delta/6$

This section proves zero-freeness near $[0, 1]$ throughout $q \geq 11\Delta/6$. Above the threshold, the proof follows the full implication

$$q \geq \frac{11}{6}\Delta \implies \text{coupling independence on } [0, 1] \implies \text{zero-freeness near } [0, 1].$$

At equality, the coupling bound is uniform on each $[\delta, 1]$, while the value at zero is supplied by the known hard-colouring coupling theorem. We first develop the complex stability estimate and separator induction needed for the second implication, then verify the coupling hypotheses of Theorem 2.4 using the soft Vigoda coupling.

3.1 Stability of complex averages

The separator argument writes a normalized partition function as $\mathbb{E}_\nu[e^{h+\eta}]$, where h is a common exterior response and η is a small local perturbation. We must compare two such averages when both the law and the perturbation change. The Lipschitz bound on h will turn a coupling of the two laws into a bound on the change of the exponent. The small oscillation of h , together with the small local perturbations, will prevent complex cancellation and allow us to compare logarithms.

Lemma 3.1 (Complex averages under nearby laws). *Let $(\mathcal{X}, d)$ be a finite metric space, let ν_0, ν_1 be probability measures, and let $A \geq 0$, $\delta \in [0, 1/8]$. Suppose $h, \eta, \eta' : \mathcal{X} \rightarrow \mathbb{C}$ satisfy*

$$|h(\xi) - h(\xi')| \leq A d(\xi, \xi'), \quad (9)$$

$$\max_{\xi, \xi'} |h(\xi) - h(\xi')| \leq 1/8, \quad (10)$$

$$\|\eta\|_\infty, \|\eta'\|_\infty \leq \delta, \quad (11)$$

where $\|f\|_\infty = \max_\xi |f(\xi)|$. The logarithm is defined by analytic continuation from $h = \eta = \eta' = 0$; the proof shows that both averages remain nonzero, and

$$\left| \text{Log} \frac{\mathbb{E}_{\nu_0}[e^{h+\eta}]}{\mathbb{E}_{\nu_1}[e^{h+\eta'}]} \right| \leq 2A W_{1,d}(\nu_0, \nu_1) + 4\delta. \quad (12)$$

Proof. First subtract $h(\xi_*)$ for a fixed $\xi_* \in \mathcal{X}$. This multiplies both averages by the same nonzero factor, so it changes neither their ratio nor whether they vanish. Keeping the notation h for the centred function, (10) now gives $\|h\|_\infty \leq 1/8$.

Choose an optimal coupling (X, Y) , with $X \sim \nu_0$ and $Y \sim \nu_1$, so that $\mathbb{E}[d(X, Y)] = W_{1,d}(\nu_0, \nu_1)$. For $0 \leq t \leq 1$, interpolate the exponents on this common probability space:

$$u_t = t(h(X) + \eta(X)) + (1-t)(h(Y) + \eta'(Y)), \quad M(t) = \mathbb{E}[e^{u_t}].$$

The marginals identify the endpoints as

$$M(0) = \mathbb{E}_{\nu_1}[e^{h+\eta'}], \quad M(1) = \mathbb{E}_{\nu_0}[e^{h+\eta}].$$

Thus the required ratio is $M(1)/M(0)$.

We first check that this path stays away from zero. Each endpoint exponent has absolute value at most $1/8 + \delta \leq 1/4$, so the same is true of u_t . The power-series inequality $|e^w - 1| \leq e^{|w|} - 1$ yields

$$|M(t) - 1| \leq \mathbb{E}[|e^{u_t} - 1|] \leq e^{1/4} - 1 < 1, \quad |M(t)| \geq 2 - e^{1/4}.$$

Thus the path lies in the disk $D := \{w \in \mathbb{C} : |w - 1| < 1\}$, and both endpoint averages are nonzero.

Next we bound how fast the average moves. The state space is finite, so differentiation under the expectation gives

$$M'(t) = \mathbb{E}[e^{u_t}(h(X) - h(Y) + \eta(X) - \eta'(Y))].$$

The Lipschitz condition bounds the first difference by $A d(X, Y)$, and the two perturbations contribute at most 2δ . Since $|e^{u_t}| \leq e^{1/4}$, we obtain

$$|M'(t)| \leq e^{1/4} \mathbb{E}[A d(X, Y) + 2\delta] = e^{1/4}(A W_{1,d}(\nu_0, \nu_1) + 2\delta).$$

This is where the coupling cost enters the estimate.

Finally, let L be the analytic logarithm on D , normalized by $L(1) = 0$. The chain rule and the fundamental theorem of calculus give

$$L(M(1)) - L(M(0)) = \int_0^1 \frac{M'(t)}{M(t)} dt.$$

This difference is the logarithm specified in the lemma: scaling h, η, η' together from zero to their given values keeps both averages in D , starting at 1. Combining the derivative bound with the lower bound on $|M(t)|$, we conclude that

$$\begin{aligned} \left| \text{Log} \frac{M(1)}{M(0)} \right| &\leq \int_0^1 \frac{|M'(t)|}{|M(t)|} dt \\ &\leq \frac{e^{1/4}}{2 - e^{1/4}} (A W_{1,d}(\nu_0, \nu_1) + 2\delta) \\ &\leq 2A W_{1,d}(\nu_0, \nu_1) + 4\delta, \end{aligned}$$

where the last step uses $e^{1/4}/(2 - e^{1/4}) < 2$. □

In the diagram below, each arrow label bounds the absolute logarithmic ratio of its endpoint averages. The horizontal solid arrow uses $\eta = \eta' = 0$. Along a vertical arrow the law stays fixed, so we take $X = Y$, and one endpoint has zero perturbation. The perturbation difference in the derivative bound is then at most δ , giving 2δ . The dashed arrow is (12). Thus coupling independence enters through the horizontal Wasserstein term, while the local perturbations account for the two vertical terms.

$$\begin{array}{ccc} \mathbb{E}_{\nu_0}[e^h] & \xleftarrow{2A W_{1,d}(\nu_0, \nu_1)} & \mathbb{E}_{\nu_1}[e^h] \\ \downarrow 2\delta & & \uparrow 2\delta \\ \mathbb{E}_{\nu_0}[e^{h+\eta}] & \xleftarrow{2A W_{1,d}(\nu_0, \nu_1) + 4\delta} & \mathbb{E}_{\nu_1}[e^{h+\eta'}]. \end{array}$$

3.2 From coupling independence to zero-freeness

We now prove Theorem 2.4. Away from 0, the proof is a direct recursion with Lemma 3.1: induction makes the exterior response Lipschitz on a separator shell, and coupling independence compares the two shell averages. The setup below only verifies these hypotheses uniformly. At 0, some inside weights vanish, so one additional additive-error estimate is needed.

3.2.1 Positive-temperature estimates

The estimates in this subsection are uniform on every compact interval $[\delta, 1] \subset (0, 1]$. Fix $G \in \mathcal{G}$, a pinning τ , and a free vertex v . All graph distances in this subsection are measured in the free graph G^τ . For $r \geq 0$, write

$$B_r(v) := \{u \in V^\tau : \text{dist}_{G^\tau}(u, v) \leq r\}, \quad S_r(v) := \{u \in V^\tau : \text{dist}_{G^\tau}(u, v) = r\}.$$

For colours a, b and a real base x , let $\nu_a^{(r)}$ and $\nu_b^{(r)}$ be the $S_r(v)$ -marginals of $\mu_{G,x}^{\tau_{v=a}}$ and $\mu_{G,x}^{\tau_{v=b}}$. If the two pinned laws admit a coupling of expected Hamming cost at most C , then, for every integer $L \geq 1$,

$$\sum_{r=2}^{L+1} W_{1,\text{Ham}}(\nu_a^{(r)}, \nu_b^{(r)}) \leq C. \quad (13)$$

Indeed, the shells are disjoint and projection cannot increase Wasserstein distance. Hence one of these L shells has cost at most C/L . If a shell is empty, the component of v has size bounded in terms of (Δ, L) and is treated as a finite local system.

Choose $r \geq 2$. For $c \in [q]$ and $\xi : S_r(v) \rightarrow [q]$, put

$$S := S_r(v), \quad U := B_{r-1}(v) \setminus \{v\},$$

and define

$$G' := G - B_{r-1}(v), \quad \tau_\xi := \tau \cup \xi.$$

Split a colouring at S , assigning to the exterior every weight factor involving a free vertex outside $B_r(v)$. Let $D_{c,\xi}(z)$ be the sum of the remaining factors over colourings of U , with $v = c$ and $S = \xi$. Since no edge crosses directly from U to the exterior,

$$\widehat{Z}_G^{\tau_{v=c}}(z) = \sum_{\xi: S \rightarrow [q]} D_{c,\xi}(z) \widehat{Z}_{G'}^{\tau_\xi}(z). \quad (14)$$

At a real base x , the shell marginal is

$$\nu_c(\xi) := \frac{D_{c,\xi}(x) \widehat{Z}_{G'}^{\tau_\xi}(x)}{\widehat{Z}_G^{\tau_{v=c}}(x)}. \quad (15)$$

The exterior graph and pinning are independent of the root colour c . Changing one coordinate of ξ changes only one pinned colour in this common exterior instance. This is precisely where the inductive one-vertex quotient bound will be used.

Lemma 3.2 (Positive-base Potts induction). *Fix integers q, Δ with $\Delta \geq 2$, let $\delta \in (0, 1]$ and $C \in [0, \infty)$, and let $\mathcal{G} \subseteq \mathcal{G}_\Delta$ be closed under taking induced subgraphs and satisfy C -coupling independence on $[\delta, 1]$. There are $\alpha, \varepsilon > 0$, depending only on (q, Δ, δ, C) , such that for every $G \in \mathcal{G}$, every pinning τ , every $x \in [\delta, 1]$, and every $|z - x| < \varepsilon$,*

$$\widehat{Z}_G^\tau(z) \neq 0,$$

and, for every $v \in V^\tau$ and $a, b \in [q]$,

$$\left| \text{Log} \frac{\widehat{Z}_G^{\tau^{v=a}}(z)/\widehat{Z}_G^{\tau^{v=a}}(x)}{\widehat{Z}_G^{\tau^{v=b}}(z)/\widehat{Z}_G^{\tau^{v=b}}(x)} \right| \leq \alpha. \quad (16)$$

The logarithm is continued from $z = x$.

Proof. Enlarging C if necessary, assume $C \geq 1$, and take $L = \lceil 64C \rceil$. Set

$$s := \Delta(\Delta - 1)^L, \quad N := 1 + \Delta \sum_{k=0}^{L-1} (\Delta - 1)^k.$$

Indeed, for $2 \leq r \leq L+1$, the breadth-first bounds give $|S_r(v)| \leq \Delta(\Delta-1)^{r-1} \leq s$ and $|B_{r-1}(v)| \leq N$. Choose

$$\alpha \leq \min \left\{ \frac{\pi}{16}, \frac{1}{16s} \right\}.$$

Here L provides enough disjoint shells to reduce the coupling cost, s bounds a shell, N separates the bounded-component case, and α is the logarithmic and angular budget. Only finitely many inside polynomials and components of size at most N occur, up to their pinned boundary data: each local system has at most $N + s$ relevant vertices, and every boundary-count coordinate lies in $\{0, \dots, \Delta\}$. They are positive on the compact interval $[\delta, 1]$. We may therefore choose $\varepsilon > 0$, uniformly in $x \in [\delta, 1]$, so that they remain nonzero for $|z - x| < \varepsilon$, the normalized child quotients of components of size at most N satisfy (16), and

$$\left| \text{Log} \frac{D_{c,\xi}(z)}{D_{c,\xi}(x)} \right| \leq \frac{\alpha}{8}, \quad \left| \Im \text{Log} \frac{z^m}{x^m} \right| + \alpha < \frac{\pi}{2} \quad (0 \leq m \leq \Delta). \quad (17)$$

All logarithms continue from $z = x$.

Induct on the number of free vertices, proving zero-freeness and (16) simultaneously. Fix G, τ, v, a, b . If the free component of v has size at most N , cancel the other components; the uniform local choice of ε gives the quotient bound directly. If it is larger, (13) supplies a shell S with

$$W_{1,\text{Ham}}(\nu_a, \nu_b) \leq C/L.$$

Induction on the common exterior defines

$$h_\xi := \text{Log} \frac{\widehat{Z}_{G'}^{\tau_\xi}(z)}{\widehat{Z}_{G'}^{\tau_\xi}(x)}.$$

When ξ and ξ' differ at one shell vertex w , regard the other shell vertices as already pinned and temporarily unpin w . The two resulting normalized laws are exactly the two one-vertex pinnings in the common exterior instance; pinned-only factors are omitted by (5). Thus the inductive quotient bound applies. Both $h_\xi - h_{\xi'}$ and that quotient logarithm are analytic and vanish at $z = x$, so they are the same continued branch. Summing along a Hamming path therefore gives

$$|h_\xi - h_{\xi'}| \leq \alpha \text{Ham}(\xi, \xi') \leq s\alpha \leq \frac{1}{16}.$$

The bounded inside has normalized change

$$\eta_c(\xi) := \text{Log} \frac{D_{c,\xi}(z)}{D_{c,\xi}(x)}, \quad \|\eta_c\|_\infty \leq \frac{\alpha}{8}.$$

The separator identity is now exactly

$$\frac{\widehat{Z}_G^{\tau^{v=c}}(z)}{\widehat{Z}_G^{\tau^{v=c}}(x)} = \mathbb{E}_{\nu_c}[e^{h+\eta_c}]. \quad (18)$$

Thus Lemma 3.1 gives

$$\left| \text{Log} \frac{\widehat{Z}_G^{\tau^{v=a}}(z)/\widehat{Z}_G^{\tau^{v=a}}(x)}{\widehat{Z}_G^{\tau^{v=b}}(z)/\widehat{Z}_G^{\tau^{v=b}}(x)} \right| \leq 2\alpha C/L + \frac{\alpha}{2} < \alpha.$$

To deduce nonvanishing of the unconditioned partition function, put

$$R_c := \frac{\widehat{Z}_G^{\tau^{v=c}}(z)}{\widehat{Z}_G^{\tau^{v=c}}(x)}$$

and choose any $a_0 \in [q]$. The colour recursion gives

$$\widehat{Z}_G^{\tau}(z) = R_{a_0} \sum_a x^{b_v^{\tau}(a)} \widehat{Z}_G^{\tau^{v=a}}(x) \exp \left(\text{Log} \frac{z^{b_v^{\tau}(a)}}{x^{b_v^{\tau}(a)}} + \text{Log} \frac{R_a}{R_{a_0}} \right).$$

The coefficients are positive. By (17) and the quotient bound, every exponential has positive real part, so the sum lies in an open half-plane and is nonzero. This closes the induction. $\square$

3.2.2 Hard-endpoint estimates

At $z = 0$, a boundary weight may vanish. Hard-feasible colourings form a positive sector, while every defective term is divisible by z .

Lemma 3.3 (One-vertex comparison at $z = 0$). *Assume $q \geq \Delta + 1$. There is $H_* = H_*(q, \Delta)$ such that, for every $G \in \mathcal{G}_\Delta$, every pinning τ , every $y \in V^\tau$, and all colours d, e ,*

$$\widehat{Z}_G^{\tau^{y=d}}(0) \leq H_* \widehat{Z}_G^{\tau^{y=e}}(0). \quad (19)$$

Proof. Measure distance in G^τ , condition on $S_2(y)$, and cancel free components outside that of y . For each shell colouring compatible with the exterior hard constraints, the exterior factor is the same for the two colours at y . Only the at most Δ vertices in $S_1(y)$ remain to be coloured. Fix such a shell colouring ξ and a root colour $r \in \{d, e\}$. For every $u \in S_1(y)$, after forbidding r and the colours used by its neighbours in $S_2(y)$, the remaining list has size at least

$$\begin{aligned} |L^\tau(u)| - 1 - |N_{G^\tau}(u) \cap S_2(y)| &\geq \deg_{G^\tau}(u) + 1 - 1 - |N_{G^\tau}(u) \cap S_2(y)| \\ &= \deg_{G^\tau[S_1(y)]}(u) + 1. \end{aligned}$$

Here the equality uses that every $u \in S_1(y)$ is adjacent to y , and its other neighbours lie in $S_1(y) \cup S_2(y)$. Greedy list-colouring of $G^\tau[S_1(y)]$ therefore gives at least one extension for each of the two root colours. There are at most $q^{|S_1(y)|} \leq q^\Delta$ extensions. Comparing term by term over the shell proves the claim with $H_* = q^\Delta$. $\square$

Proposition 3.4 (Potts disk around $z = 0$). *Fix integers q, Δ with $\Delta \geq 2$, $q \geq \Delta + 1$, let $C_0 \in [0, \infty)$, and let $\mathcal{G} \subseteq \mathcal{G}_\Delta$ be closed under taking induced subgraphs and satisfy C_0 -coupling independence at $x = 0$. There is $\rho_0 = \rho_0(q, \Delta, C_0) > 0$ such that*

$$\widehat{Z}_G^{\tau}(z) \neq 0 \quad (|z| < \rho_0)$$

for every $G \in \mathcal{G}$ and every pinning τ .

Proof. Enlarging C_0 if necessary, assume $C_0 \geq 1$, put $L = \lceil 64C_0 \rceil$, and set

$$s := \Delta(\Delta - 1)^L, \quad N := 1 + \Delta \sum_{k=0}^{L-1} (\Delta - 1)^k.$$

For every vertex v and $2 \leq r \leq L + 1$, these quantities bound $|S_r(v)|$ and $|B_{r-1}(v)|$, respectively. Choose

$$\alpha \leq \min \left\{ \frac{\pi}{16}, \frac{1}{16s} \right\}.$$

These constants have the same roles as above: L yields a low-cost shell, s bounds its Hamming diameter, N isolates bounded components, and α is the logarithmic budget. The only new step is to separate the hard-feasible contribution from an additive error divisible by z . Below K denotes a positive constant depending only on (q, Δ, C_0) , and may increase from line to line. We prove, for a sufficiently small uniform ρ_0 , zero-freeness and

$$\left| \text{Log} \frac{\widehat{Z}_G^{\tau^{v=a}}(z)/\widehat{Z}_G^{\tau^{v=a}}(0)}{\widehat{Z}_G^{\tau^{v=b}}(z)/\widehat{Z}_G^{\tau^{v=b}}(0)} \right| \leq \alpha \quad (20)$$

for every $G \in \mathcal{G}$, every pinning τ , every $v \in V^\tau$, and all a, b . The denominators are positive by Lemma 2.2. Throughout this proof, endpoint logarithms are the analytic branches continued from $z = 0$ and normalized there by $\text{Log} 1 = 0$. Induct on the number of free vertices. In the current inductive instance, fix G, τ, v , measure distance in G^τ , and use the sets $B_r(v), S_r(v)$ and the separator data $S, U, G', \tau_\xi, D_{c,\xi}$ from (14)–(15), now at the real base $x = 0$.

Components of size at most N are handled uniformly by finiteness and continuity at $z = 0$. For a larger component, choose a shell with $W_{1,\text{Ham}}(\nu_a, \nu_b) \leq C_0/L$. Induction on the common exterior defines

$$h_\xi := \text{Log} \frac{\widehat{Z}_{G'}^{\tau_\xi}(z)}{\widehat{Z}_{G'}^{\tau_\xi}(0)},$$

and, for a one-coordinate change at a shell vertex w , regard all other shell vertices as part of the base pinning and temporarily unpin w . This realizes the two normalized exterior laws as the two one-vertex pinnings of the same induced instance; pinned-only factors are omitted by (5). Hence the inductive response bound (20) applies to this smaller instance. Branch uniqueness at $z = 0$ then identifies $h_\xi - h_{\xi'}$ with the inductive one-vertex quotient logarithm. Summing along a Hamming path gives

$$|h_\xi - h_{\xi'}| \leq \alpha \text{Ham}(\xi, \xi') \leq s\alpha \leq \frac{1}{16}.$$

Here ν_c is the shell marginal at $x = 0$. For each c , choose a hard-feasible colouring of the free vertices under $\tau^{v=c}$, and let ξ_c be its shell restriction. This witness supplies at least one hard inside extension, so

$$D_{c,\xi_c}(0) \geq 1, \quad \widehat{Z}_G^{\tau^{v=c}}(0) \geq \widehat{Z}_{G'}^{\tau_{\xi_c}}(0).$$

For a fixed inside colouring φ , its contribution is $z^{m(\varphi)}$ for some $m(\varphi) \geq 0$. After subtracting the value at 0, this contribution is zero if $m(\varphi) = 0$ and is $z^{m(\varphi)}$ otherwise. Since there are at most q^N inside colourings,

$$|D_{c,\xi}(z) - D_{c,\xi}(0)| \leq q^N |z| \quad (|z| \leq 1/2).$$

Splitting $D_{c,\xi}(z)$ into its value at 0 and its error in the separator identity gives

$$R_c := \frac{\widehat{Z}_G^{\tau^{v=c}}(z)}{\widehat{Z}_G^{\tau^{v=c}}(0)} = M_c + E_c, \quad M_c := \mathbb{E}_{\nu_c}[e^h],$$

where, explicitly,

$$E_c = \frac{\sum_{\xi} (D_{c,\xi}(z) - D_{c,\xi}(0)) \widehat{Z}_{G'}^{\tau_{\xi}}(0) e^{h_{\xi}}}{\widehat{Z}_G^{\tau^{v=c}}(0)}.$$

Changing the shell colours one at a time and applying Lemma 3.3 at most s times gives

$$\widehat{Z}_{G'}^{\tau_{\xi}}(0) \leq H_*^s \widehat{Z}_{G'}^{\tau_{\xi c}}(0).$$

There are at most q^s shell colourings, and $|h_{\xi} - h_{\xi c}| \leq 1/16$. The preceding bounds therefore give

$$|E_c| \leq q^{N+s} H_*^s e^{1/16} |z| |e^{h_{\xi c}}|.$$

For the main term,

$$\begin{aligned} \left| \frac{M_c}{e^{h_{\xi c}}} - 1 \right| &\leq \sum_{\xi} \nu_c(\xi) |e^{h_{\xi} - h_{\xi c}} - 1| \\ &\leq e^{1/16} - 1 < \frac{1}{4}. \end{aligned}$$

Hence

$$|M_c| \geq (2 - e^{1/16}) |e^{h_{\xi c}}| > \frac{3}{4} |e^{h_{\xi c}}|.$$

After increasing K , these two estimates give $|E_c/M_c| \leq K|z|$. Shrink ρ_0 , if necessary, so that $\rho_0 \leq 1/2$, $K\rho_0 \leq 1/2$, and $4K\rho_0 < \alpha/2$. Then

$$|\text{Log}(1 + E_c/M_c)| \leq 2K|z|,$$

with the logarithm continued from $z = 0$.

Apply Lemma 3.1 with $A = \alpha$ and $\eta = \eta' = 0$. It gives

$$\left| \text{Log} \frac{M_a}{M_b} \right| \leq 2\alpha C_0/L.$$

We have

$$\text{Log} \frac{R_a}{R_b} = \text{Log} \frac{M_a}{M_b} + \text{Log}(1 + E_a/M_a) - \text{Log}(1 + E_b/M_b),$$

Combining this identity with $L \geq 64C_0$ and $4K|z| < \alpha/2$, we find that the absolute value of its right-hand side is at most $\alpha/32 + \alpha/2 < \alpha$. This proves (20).

To close the induction, the parent recursion is

$$\widehat{Z}_G^{\tau}(z) = \sum_c z^{b_v^{\tau}(c)} \widehat{Z}_G^{\tau^{v=c}}(z).$$

Choose a hard-feasible colouring σ and put $a_0 = \sigma_v$, so $b_v^{\tau}(a_0) = 0$. After division by R_{a_0} , the terms with $b_v^{\tau}(c) = 0$ have positive real part. The a_0 -term is exactly $\widehat{Z}_G^{\tau^{v=a_0}}(0)$, so their sum has real part at least this value. The remaining terms are divisible by z . Here $|R_c/R_{a_0}| \leq e^{\alpha}$ and $\widehat{Z}_G^{\tau^{v=c}}(0) \leq H_* \widehat{Z}_G^{\tau^{v=a_0}}(0)$, so Lemma 3.3 bounds the total modulus of the at most q remaining terms by $K|z| \widehat{Z}_G^{\tau^{v=a_0}}(0)$. A final shrink of ρ_0 makes this error smaller than the hard-feasible contribution and closes the induction. $\square$

Proof of Theorem 2.4. Let ρ_0 come from Proposition 3.4, set

$$\delta_0 := \min\{1/2, \rho_0/4\},$$

and apply Lemma 3.2 with $\delta = \delta_0$ and $C = C_{\delta_0}$. Let ε_+ be the resulting uniform radius and put $\varepsilon := \min\{\varepsilon_+, \delta_0\}$. If a nearest point $x \in [0, 1]$ satisfies $x \leq \delta_0$, then $|z| < 2\delta_0 < \rho_0$; otherwise $x \in [\delta_0, 1]$ and the positive-base induction applies. The relation between the pinned and normalized functions is (3). $\square$

3.3 Proof of the Potts zero-free theorem

Proof of Theorem 1.1. The hypotheses imply $q \geq \Delta + 2$. If $q > 11\Delta/6$, then Theorem 3.13, proved in the final subsection, gives one coupling-independence constant on all of $[0, 1]$. If $q = 11\Delta/6$, then Corollary 3.17, also proved there, supplies the two coupling inputs required by Theorem 2.4. Applying Theorem 2.4 to $\mathcal{G}_\Delta$ gives the normalized zero-free statement. The assertion for the unnormalized pinned polynomial follows from (3). $\square$

3.4 Soft flip dynamics and comparison tools

To compare the laws with two different root colours, we compare each to the common system obtained by deleting the root. Pinning adds at most Δ boundary constraints. We need two estimates: how much these constraints change one update, and how strongly the kernel contracts. Stationary comparison converts them into a distance between stationary laws; the triangle inequality then gives coupling independence. This subsection constructs the kernels and bounds the boundary effect. Sections 3.5.1 and 3.5.2 supply contraction, and continuity covers $x = 0$.

3.4.1 Stationary comparison and endpoint continuity

For a Markov kernel P , write νP for the law after one update from ν . We use the following form of [CFG⁺25, Lemma 19].

Lemma 3.5 (Stationary comparison). *Let d be the shortest-path metric of a finite connected state graph with positive edge lengths. Let P, Q have stationary laws ν_P, ν_Q . Suppose $0 < \alpha \leq 1$, $C \geq 0$, and*

$$W_{1,d}(P(s, \cdot), P(t, \cdot)) \leq (1 - \alpha)d(s, t) \quad \text{for adjacent } s, t, \quad \mathbb{E}_{s \sim \nu_Q} W_{1,d}(P(s, \cdot), Q(s, \cdot)) \leq C.$$

Then $W_{1,d}(\nu_P, \nu_Q) \leq C/\alpha$.

Proof. Summing along shortest paths extends the first bound to all states; coupling the initial states then gives $W_{1,d}(\nu P, \mu P) \leq (1 - \alpha)W_{1,d}(\nu, \mu)$. Stationarity and the second hypothesis yield

$$W_{1,d}(\nu_P, \nu_Q) \leq W_{1,d}(\nu_P P, \nu_Q P) + W_{1,d}(\nu_Q P, \nu_Q Q) \leq (1 - \alpha)W_{1,d}(\nu_P, \nu_Q) + C.$$

Rearrange. $\square$

Lemma 3.6 (Finite-state continuity of normalized laws). *On a finite space Ω , let two families of nonnegative weights be continuous at $x = 0$, with positive total weights there. Their normalized laws $\mu_{0,x}, \mu_{1,x}$ converge in total variation as $x \downarrow 0$, and for every metric d on Ω ,*

$$W_{1,d}(\mu_{0,x}, \mu_{1,x}) \longrightarrow W_{1,d}(\mu_{0,0}, \mu_{1,0}).$$

Thus a uniform positive-activity coupling bound extends to $x = 0$.

Proof. The normalized weights converge pointwise. Use $W_{1,d}(\nu, \nu') \leq \text{diam}_d(\Omega) \|\nu - \nu'\|_{\text{TV}}$ and the triangle inequality. $\square$

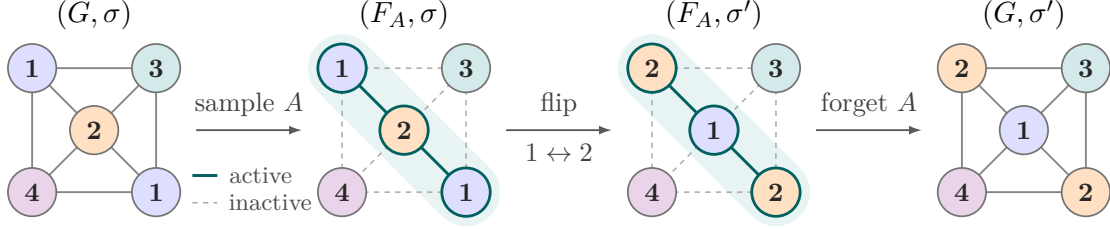


Figure 1: One soft flip update, without pins. The highlighted $\{1, 2\}$ -component is flipped on the sampled hard instance F_A . Discarding A returns the same updated colouring σ' to G .

3.4.2 Random hard constraints and the soft flip kernel

Fix $G \in \mathcal{G}_\Delta$, a pinning τ , and $x \in (0, 1]$. Let $\mathcal{C}_\tau$ consist of the free edges and the individually labelled free–pinned edges; pinned-only edges have already been removed by normalization. Each member imposes an inequality between its endpoint colours. The identity

$$x\mathbf{1}_{[s=t]} + \mathbf{1}_{[s \neq t]} = x + (1 - x)\mathbf{1}_{[s \neq t]} \quad (21)$$

gives the following representation. Given σ , activate each satisfied inequality independently with probability $1 - x$. For the active set A , let E_A contain its free edges and put

$$L_{A,u} = [q] \setminus \{\tau_p : up \in A \text{ is a free–pinned edge}\}, \quad F_A = (V^\tau, E_A, L_A).$$

For any hard list instance $F = (W, E, L)$, write Ω_F for its proper colourings respecting every list.

Lemma 3.7 (Soft Potts as a random hard-colouring model). *If $\sigma \sim \mu_{G,x}^\tau$ and A is sampled as above, then σ , conditional on A , is uniform on Ω_{F_A} .*

Proof. For $M = |\mathcal{C}_\tau|$, the joint unnormalized weight is

$$x^{M-|A|}(1-x)^{|A|}\mathbf{1}_{[\sigma \in \Omega_{F_A}]}. \quad (22)$$

Summing over A gives a factor 1 for each satisfied constraint and x for each unsatisfied one, hence the Potts weight. For fixed A , the displayed weight is constant on Ω_{F_A} . $\square$

A flip profile $\rho = (\rho_s)_{s \geq 0}$ satisfies $\rho_0 = 0$ and $0 \leq \rho_s \leq 1$.

Definition 3.8 (Hard list-flip transition). From $X \in \Omega_F$, choose (u, c) uniformly in $W \times [q]$. If $c = X_u$, hold. Otherwise, let $S_X(u, c)$ be the component of u induced by colours X_u, c . Swap its two colours with probability $\rho_{|S_X(u, c)|}/|S_X(u, c)|$, provided the swap respects every list; otherwise hold. Denote this kernel by Φ_F^ρ , taking the identity kernel when $W = \emptyset$.

A feasible component swap of size s has s proposals, hence total probability $\rho_s/(|W|q)$. The swap is an involution with the same reverse rate, so Φ_F^ρ is reversible for the uniform law.

Definition 3.9 (Soft profile kernel). The kernel $K_{x,\rho}^{G,\tau}$ samples A conditional on the current colouring, applies $\Phi_{F_A}^\rho$, and returns the updated colouring to G , keeping τ fixed.

Lemma 3.10 (Soft flip kernels). *For every profile ρ and $x \in (0, 1]$, $K_{x,\rho}^{G,\tau}$ is reversible with stationary law $\mu_{G,x}^\tau$. If $\rho_1 > 0$, it is irreducible on $[q]^{V^\tau}$.*

Proof. Conditional detailed balance for each A , multiplied by its marginal probability in (22) and summed over A , gives detailed balance for the spin marginal. The empty active set has positive probability when $x > 0$; on this event, $\rho_1 > 0$ permits every one-site recolouring. $\square$

3.4.3 Common activations and boundary sensitivity

For adjacent colourings X, Y differing at v , with $X_v = a \neq b = Y_v$, draw one independent Bernoulli($1 - x$) coin per labelled constraint. Activate that constraint on either side exactly when the coin is one and the inequality is satisfied there. Write $\mathcal{A} = (A_X, A_Y)$ for the outcome and F_X, F_Y for the hard instances.

Lemma 3.11 (The two hard instances differ only near v). *The common-coin construction has the correct activation marginals. The two instances have the same induced graph and lists off v . Their incident edge sets can differ only at neighbours of colour a or b , and their root lists can differ only in those two colours.*

Proof. An inequality has different satisfaction status only if it is incident to v and its other endpoint has colour a or b . $\square$

Lemma 3.12 (Effect of adding one labelled free–pinned edge). *Let two pinned systems have the same free vertex set W , of size $m \geq 1$, and the same labelled constraints except for one additional free–pinned edge in the plus system. For a profile with $R_\rho := \sup_{s \geq 1} s\rho_s < \infty$, their soft kernels satisfy*

$$W_{1,\text{Ham}}(K_-(\sigma, \cdot), K_+(\sigma, \cdot)) \leq \frac{(1-x)R_\rho}{mq} (\sigma \in [q]^W). \quad (23)$$

For hard list kernels, deleting one colour from one list gives the bound $R_\rho/(mq)$ on colourings feasible for the smaller list. Adding k labelled constraints multiplies the respective bound by k .

Proof. Couple the common activation coins and flip proposals. Suppose the extra constraint forbids colour a at u . It is inactive when $\sigma_u = a$; otherwise it is active with probability $1 - x$. When active, it can change acceptance only for the unique $\{\sigma_u, a\}$ -component containing u . If this component has size s , couple its rejected plus move to its minus move: the cost is at most $s\rho_s/(mq) \leq R_\rho/(mq)$. Average over activation to prove (23); deterministic list deletion omits the factor $1 - x$. The triangle inequality handles multiple labelled constraints. $\square$

3.5 Coupling independence at and above the Vigoda threshold

For $\Delta \geq 3$, the hard-colouring coupling input follows from [CDM⁺19, Liu21, BCC⁺22] and is recorded explicitly in [CFG⁺25, Theorem 20]. We extend it to soft activities by conditioning on the random hard constraints of Section 3.4. The only local estimate needed is Proposition 3.14; averaging and stationary comparison then give the following theorem.

Theorem 3.13 (Hamming coupling independence on $[0, 1]$). *Let q, Δ be positive integers with $\Delta \geq 2$ and $q > 11\Delta/6$. Put*

$$\gamma_0 := \frac{q}{\Delta} - \frac{11}{6} > 0.$$

Then, for every $G \in \mathcal{G}_\Delta$, every pinning τ , every $x \in [0, 1]$, every $r \in V^\tau$, and all $a, b \in [q]$,

$$W_{1,\text{Ham}}(\mu_{G,x}^{\tau^{r=a}}, \mu_{G,x}^{\tau^{r=b}}) \leq \frac{2(1-x)\Delta}{q - (11/6)(1-x)\Delta} \leq \frac{2}{\gamma_0}.$$

The two laws are on $V^\tau \setminus \{r\}$. At $x = 0$ they are well defined by Lemma 2.2; indeed, the hypotheses imply $q \geq \Delta + 2$.

Use Vigoda's profile [Vig00]

$$p_0 = 0, \quad p_1 = 1, \quad p_2 = \frac{13}{42}, \quad p_3 = \frac{1}{6}, \quad p_4 = \frac{2}{21}, \quad p_5 = \frac{1}{21}, \quad p_6 = \frac{1}{84}, \quad p_s = 0 \quad (s \geq 7), \quad (24)$$

and abbreviate $\Phi_F := \Phi_F^p$ and $K_x^{G,\tau} := K_{x,p}^{G,\tau}$.

3.5.1 Conditional hard estimate

Fix $G \in \mathcal{G}_\Delta$, a pinning τ , and $x \in (0, 1]$. For adjacent colourings X, Y with $X_v = a \neq b = Y_v$, condition on their common-coin activation outcome $\mathcal{A}$. Write F_X, F_Y for the resulting hard instances, $n = |V^\tau|$, and G_0 for their common induced graph off v , with its common colouring. The two current colourings are feasible, so $a \in L_{X,v}$ and $b \in L_{Y,v}$; the root lists agree on every other colour. Put

$$N_c = \{u \sim_{G^\tau} v : X_u = Y_u = c, \text{ } vu \text{ is active on at least one side}\}, \quad \ell = |L_{X,v} \cap L_{Y,v}|.$$

Proposition 3.14 (Conditional hard estimate). *The updates $\Phi_{F_X}(X, \cdot)$ and $\Phi_{F_Y}(Y, \cdot)$ admit a coupling such that*

$$nq \mathbb{E}[\text{Ham}(X', Y') - 1 \mid \mathcal{A}] \leq \frac{11}{6} \sum_c |N_c| - \ell. \quad (25)$$

Proof. We first match component moves and then bound the Hamming drift separately for each colour. Detailed size estimates are needed only when $c \notin \{a, b\}$ and $|N_c| = 1, 2$; the other cases admit coarse bounds.

Scale all move probabilities by nq . Write $p(M)$ for the scaled mass of a component move: it is $p_{|M|}$ if the swap is feasible and zero otherwise. A root component contains v ; an off-root component does not. Match every move that is off-root in both instances identically.

Call $c \notin \{a, b\}$ a regular colour. Every active edge from v to N_c is then common to both instances. Let $\mathcal{C}$ be the family of $\{a, c\}$ -components of G_0 meeting N_c , and let $\mathcal{D}$ be their $\{b, c\}$ -counterparts. Adding v joins each family into a root component on one side; on the other side its members remain off-root. The remaining moves are:

colour pair	moves in X	moves in Y
$\{a, c\}$	root R	off-root $C \in \mathcal{C}$
$\{b, c\}$	off-root $D \in \mathcal{D}$	root S

Here

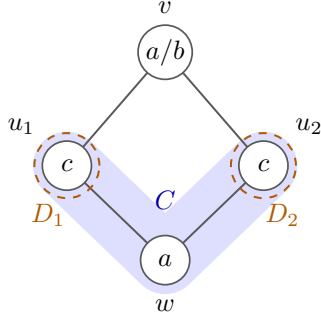
$$R = \{v\} \cup \bigcup_{C \in \mathcal{C}} C, \quad S = \{v\} \cup \bigcup_{D \in \mathcal{D}} D.$$

A root swap is feasible exactly when its new root colour is allowed and all its constituent off-root swaps are feasible: these are precisely its vertexwise list tests. This equivalence will also handle blocked moves in the estimates below.

Matching the moves. Write $m = |N_c|$. If $m = 0$, match the two singleton root moves, of mass one, when c is in both root lists; otherwise both are blocked. If $m \geq 1$, choose largest components $C_\star, D_\star$, breaking ties by the least-indexed neighbour in a fixed ordering of N_c . Match the whole mass $p(R)$ of R to $C_\star$, and the whole mass $p(S)$ of S to $D_\star$. Root feasibility and the monotonicity of p_s ensure that these allocations fit. The residual off-root masses are

$$r_C = p(C) - \mathbf{1}_{[C=C_\star]} p(R), \quad r_D = p(D) - \mathbf{1}_{[D=D_\star]} p(S).$$

Shared off-root component



Root matches and incidence allocation

$$R_X \longleftrightarrow C_Y : 8, \quad (D_1)_X \longleftrightarrow S_Y : 14.$$

Move	Initial	After root matching	Used at u_1	Used at u_2	Left
C_Y	14	6	3	3	0
$(D_1)_X$	84	70	3	0	67
$(D_2)_X$	84	84	0	3	81

$$u_1 : \min\{6/2, 70\} = 3, \quad u_2 : \min\{6/2, 84\} = 3.$$

Figure 2: Incidence matching on a four-cycle with full lists, $q = 4$, and all four edges active. The root has colour a in X and b in Y . Off the root, $C = \{u_1, w, u_2\}$ and $D_i = \{u_i\}$, so $|R| = 4$ and $|S| = 3$; choose $D_\star = D_1$. All numbers are 84 times the scaled masses, hence represent actual probabilities in units of $1/(84nq)$. Each incidence column records both sides of the same match. The two incidences use $3 + 3 = 6$, exactly the residual mass of C .

For each $u_i \in N_c$, let C_i, D_i contain u_i , and match an additional mass

$$\min \left\{ \frac{r_{C_i}}{|C_i \cap N_c|}, \frac{r_{D_i}}{|D_i \cap N_c|} \right\} \quad (26)$$

between those two off-root moves. Dividing by the number of incidences ensures that no component's residual mass is exceeded. Figure 2 illustrates this division when a single component meets two root neighbours.

For the pair $\{a, b\}$, the Y -root consists of v and the common components meeting N_a ; the X -root uses those meeting N_b . A component meeting both sets belongs to both roots and gives no off-root move. Assign the Y -root and the X -off-root moves meeting N_a but not N_b to colour a , and define the b -group symmetrically. When $|N_a| = 1$ and its component C is off-root on the X -side, match the whole Y -root mass to C . Otherwise match it to holding. Do the symmetric construction for b . The off-root capacity follows again from feasibility and monotonicity; each holding row has mass at least n , while at most one root move of mass at most one uses it.

These groups exhaust the remaining side-labelled moves without overlap: colour pairs avoiding a, b were common, and the preceding description classifies all pairs containing either root colour. Complete the partial coupling by pairing residual row and column masses proportionally. If they are α_M, β_N , with common total $T > 0$, add $\alpha_M \beta_N / T$ to entry (M, N) ; if $T = 0$, nothing remains. All allocations have the required marginals after division by nq .

The charges. An unmatched pair of moves M, N increases Hamming distance by at most $|M| + |N|$. Charge each residual move its size times its residual mass; let Q_c be these charges plus the actual drift of the prescribed matches in group c . Common matches have zero drift, and the residual cost separates by side even when the completion joins different groups. Consequently it suffices to prove

$$nq \mathbb{E}[\text{Ham}(X', Y') - 1 \mid \mathcal{A}] \leq \sum_c Q_c, \quad (27)$$

$$Q_c \leq \frac{11}{6} |N_c| - \mathbf{1}_{[c \in L_{X,v} \cap L_{Y,v}]}. \quad (28)$$

For a regular colour, matching R to $C_\star$ creates disagreements only on the other constituents of R ; its drift is $|R| - |C_\star| - 1$. The analogous cost for S is $|S| - |D_\star| - 1$. Each off-root match in

(26) saves at least one unit because $C_i \cap D_i$ contains u_i . Adding root-match costs and residual size charges, and subtracting these matching savings, gives, for $m \geq 1$,

$$Q_c \leq (|R| - |C_\star| - 1)p(R) + (|S| - |D_\star| - 1)p(S) + \sum_C |C|r_C + \sum_D |D|r_D - \sum_{i=1}^m \min \left\{ \frac{r_{C_i}}{|C_i \cap N_c|}, \frac{r_{D_i}}{|D_i \cap N_c|} \right\}. \quad (29)$$

The following bounds follow directly from the six nonzero profile values:

$$rp_r \leq 1, \quad rp_r \leq \frac{13}{21} \ (r \geq 2), \quad (r-1)p_r \leq \frac{1}{3}, \quad (r-2)p_r \leq \frac{4}{21}. \quad (30)$$

For two off-root moves meeting the same neighbour, with sizes r, s and masses u, w , write $g_{r,s}(u, w) = ru + sw - \min\{u, w\}$ for their size charges minus the guaranteed matching saving. This function is nondecreasing in each argument for $r, s \geq 1$. Since p_r is nonincreasing, $g_{r,s}(p_r, p_s) = rp_r + (s-1)p_s$ when $r \leq s$. Using (30), with $r = 1$ or $r \geq 2$, gives

$$g_{r,s}(u/j, w/k) \leq \frac{4}{3} \quad (0 \leq u \leq p_r, \ 0 \leq w \leq p_s, \ j, k \geq 1). \quad (31)$$

Splitting each component charge among its incidences in N_c now bounds the last three terms of (29) by $4m/3$. If c is blocked at the root, both root masses vanish, proving (28). If it is allowed and $m \geq 3$, the root terms are each at most $4/21$, giving $Q_c \leq 8/21 + 4m/3 \leq 11m/6 - 1$. For $m = 0$, the singleton match gives $Q_c = -\mathbf{1}_{[c \in L_{X,v} \cap L_{Y,v}]}$.

One and two neighbours: analytic bounds. It remains to treat an allowed regular colour with $m = 1, 2$. For $m = 1$, let the two off-root sizes be r, s and put $d_r = p_r - p_{r+1}$. Root feasibility equals off-root feasibility, so the residual masses lie in $\{0, d_r\}$ and $\{0, d_s\}$, respectively. Thus (29) and monotonicity of g give $Q_c \leq g_{r,s}(d_r, d_s)$. The profile gives

$$d_1 = \frac{29}{42}, \quad rd_r \leq \frac{2}{7} \ (r \geq 2), \quad (r-1)d_r \leq \frac{1}{7} \ (r \geq 2).$$

If both sizes are at least two, the bound is at most $4/7$. If both are one, it is $29/42$; otherwise it is at most $29/42 + 1/7 = 5/6$, as required.

For $m = 2$, first suppose that some off-root component contains both neighbours. Its size is at least three, since those neighbours have the same colour. Its root term in (29) is zero, and its off-root charge is at most $rp_r \leq 1/2$ for $r \geq 3$. The other family contributes at most two in off-root charges. Its root term is zero for one component, and at most $wp_{1+u+w} \leq 1/6$ for two sizes $u \geq w$. Hence $Q_c \leq 1/2 + 2 + 1/6 = 8/3$.

Otherwise each family has two distinct components. Index them by their neighbours, with sizes r_1, r_2 and s_1, s_2 , and feasibility bits $f_i, g_i \in \{0, 1\}$. Let $P = p(R)$, $Q = p(S)$. Before allocating mass to the root matches, the off-root bound is

$$H_0 = \sum_{i=1}^2 g_{r_i, s_i}(f_i p_{r_i}, g_i p_{s_i}).$$

For $\mathcal{C}$, the root match costs $(\min_i r_i)P$, removes size charge $(\max_i r_i)P$, and loses at most P of matching saving. These three contributions explain its correction to H_0 . Adding the corresponding correction for $\mathcal{D}$ gives

$$Q_c \leq H_0 + (1 - \max r_i + \min r_i)P + (1 - \max s_i + \min s_i)Q. \quad (32)$$

A correction is nonpositive for unequal sizes. For two equal sizes $k \geq 2$, it is at most p_{2k+1} , which is nonzero only for $k = 2$ and then equals $1/21$. For two singletons it is at most $p_3 = 1/6$.

Thus positive corrections occur only within equal-size families. Singletons also have the largest off-root mass, whereas $rp_r \leq 13/21$ for every nonsingleton. This suggests classifying the four sizes by their number h of singletons. A pair of singletons has $g_{1,1} \leq 1$; all other pairs satisfy $g_{r,s} \leq 4/3$. These bounds give the following upper bounds in (32):

h	arrangement	H_0	total correction	Q_c
0	any	$52/21$	$2/21$	$18/7$
1	any	$4/3 + 26/21$	$1/21$	$55/21$
2	one singleton in each family	$2 \cdot 4/3$	0	$8/3$
3	any	$1 + 4/3$	$1/6$	$5/2$
4	any	2	$2 \cdot 1/6$	$7/3$

Every last-column entry is at most $8/3$.

It remains to handle $h = 2$ with both singletons in one family, say $\mathcal{C}$. Here the general correction bound is too coarse. If $P > 0$, both singleton swaps are feasible and their residual masses are at least $1 - p_3 = 5/6 > p_2$. Since both opposite masses are at most p_2 , subtracting P loses no matching saving. The root cost P and removed size charge P cancel, so $\mathcal{C}$ has zero correction. The $\mathcal{D}$ -correction is nonpositive for unequal sizes and zero for equal sizes at least three. Only $(s_1, s_2) = (2, 2)$ remains, where

$$Q_c \leq 2g_{1,2}(p_1, p_2) + p_5 = 2(1 + p_2) + p_5 = \frac{8}{3};$$

otherwise $H_0 \leq 8/3$ suffices. This proves the regular-colour bound entirely by the displayed scalar inequalities.

The two root colours. Consider colour a , writing $m = |N_a|$; colour b is symmetric. If $a \notin L_{Y,v}$, its root move is blocked and its at most m off-root components each cost at most $sp_s \leq 1$. If $a \in L_{Y,v}$ and $m = 0$, its singleton root/holding match coalesces, giving $Q_a = -1$. For $m \geq 2$, the root/holding match has drift at most $(s - 2)p_s \leq 4/21$, so $Q_a \leq m + 4/21 \leq 11m/6 - 1$.

In the remaining case $a \in L_{Y,v}$ and $m = 1$, its root is $\{v\} \cup C$, where C has size s . If the root swap is infeasible, the corresponding off-root swap, when present, fails the same list test, giving zero charge. If the root swap is feasible and C also meets N_b , there is no off-root move in this group and the charge is at most $(s - 1)p_{s+1} \leq 4/21$. Otherwise the root/off-root match coalesces, leaving charge at most

$$-p_{s+1} + s(p_s - p_{s+1}) = sp_s - (s + 1)p_{s+1} \leq \frac{8}{21} < \frac{5}{6}.$$

This proves (28) for all colours; summing proves the proposition. $\square$

Arithmetic check. The embedded `verifier.py` independently checks the scalar bounds and the charge in (29) for $m = 1, 2$. Run it with `python3 verifier.py`. It uses exact rational arithmetic, enumerates component partitions, sizes capped at seven, and off-root feasibility bits, and derives root feasibility from the list-test equivalence above. The analytic proof establishes these bounds without enumeration. The companion Lean development also checks the finite inequalities and their use in the concrete coupling proof; see the companion repository [Shi26].

3.5.2 Averaging and stationary comparison

Lemma 3.15 (Soft path contraction). *For $G \in \mathcal{G}_\Delta$, a pinning τ , and $x \in (0, 1]$, put $t = 1 - x$ and $n = |V^\tau| \geq 1$. Adjacent states admit a coupling satisfying*

$$nq \mathbb{E}[\text{Ham}(X', Y') - 1] \leq \frac{11}{6}t\Delta - q. \quad (33)$$

In particular, if $q > (11/6)t\Delta$, the soft kernel contracts Hamming distance by $1 - \kappa_x/n$, where

$$\kappa_x = \frac{q - (11/6)t\Delta}{q} > 0. \quad (34)$$

Proof. At the disagreement vertex, let d be its free degree, b_τ its number of pinned neighbours, and k the number of boundary constraints active on at least one side. Each incident constraint is satisfied on at least one side, so its union activation is exactly its Bernoulli(t) coin. Thus

$$\mathbb{E} \sum_c |N_c| = td, \quad \mathbb{E} k = tb_\tau, \quad \ell \geq q - k.$$

Averaging Proposition 3.14 gives

$$nq \mathbb{E}[\text{Ham}(X', Y') - 1] \leq \frac{11}{6}td + tb_\tau - q \leq \frac{11}{6}t\Delta - q.$$

Divide by nq ; shortest-path coupling extends the adjacent-state bound to all pairs. $\square$

Proposition 3.16 (Vigoda-profile coupling for newly pinned laws). *Let q, Δ be positive integers with $\Delta \geq 2$. For every $G \in \mathcal{G}_\Delta$, every pinning τ , every $r \in V^\tau$, all $a, b \in [q]$, and every $x \in (0, 1]$ satisfying $q > (11/6)(1 - x)\Delta$,*

$$W_{1, \text{Ham}}(\mu_{G, x}^{\tau^{r=a}}, \mu_{G, x}^{\tau^{r=b}}) \leq \frac{2(1 - x)\Delta}{q - (11/6)(1 - x)\Delta}. \quad (35)$$

If $q > 11\Delta/6$, the same formula holds at $x = 0$.

Proof. The cases $a = b$ or $m := |V^\tau \setminus \{r\}| = 0$ are immediate. Otherwise compare each newly pinned system to the common middle system $(G - r, \tau)$, with stationary law μ_{mid} . The child of colour c adds at most Δ labelled boundary constraints to this middle system. Since $\sup_s sp_s = 1$, Lemma 3.12 gives, for every common state σ ,

$$W_{1, \text{Ham}}(K_x^{G, \tau^{r=c}}(\sigma, \cdot), K_x^{G-r, \tau}(\sigma, \cdot)) \leq \frac{(1 - x)\Delta}{mq}. \quad (36)$$

The child kernel contracts by $1 - \kappa_x/m$. Apply Lemma 3.5, using the stationary laws from Lemma 3.10, and then the triangle inequality:

$$W_{1, \text{Ham}}(\mu_{G, x}^{\tau^{r=c}}, \mu_{\text{mid}}) \leq \frac{(1 - x)\Delta}{q\kappa_x}, \quad W_{1, \text{Ham}}(\mu_{G, x}^{\tau^{r=a}}, \mu_{G, x}^{\tau^{r=b}}) \leq \frac{2(1 - x)\Delta}{q\kappa_x}.$$

This is (35). If $q > 11\Delta/6$, both hard pinned laws are nonempty by Lemma 2.2; their polynomial weights and Lemma 3.6 pass the bound to $x = 0$. Its maximum on $[0, 1]$ is $2/\gamma_0$, proving Theorem 3.13. $\square$

Corollary 3.17 (Critical-line coupling input). *Let q, Δ be integers with $\Delta \geq 2$ and $q = 11\Delta/6$. Then $\mathcal{G}_\Delta$ satisfies coupling independence at $x = 0$ with a finite constant, and for every $\delta \in (0, 1]$ it satisfies coupling independence on $[\delta, 1]$ with constant $12/(11\delta)$.*

Proof. For $x \in [\delta, 1]$, Proposition 3.16 gives

$$W_{1,\text{Ham}}(\mu_{G,x}^{\tau^{r=a}}, \mu_{G,x}^{\tau^{r=b}}) \leq \frac{12(1-x)}{11x} \leq \frac{12}{11\delta}.$$

At $x = 0$, integrality forces Δ to be a positive multiple of 6, so [CFG⁺25, Theorem 20 and Proposition 22] applies. The conditional weights in [CFG⁺25, Section 2] omit pinned-only edges and allow improper pinnings, agreeing with our normalization. Apply its coupling bound to $\tau^{r=a}$ and $\tau^{r=b}$, then project onto $V^\tau \setminus \{r\}$, to obtain the bound required by Definition 2.3. The list-contraction input to that theorem originates in [CDM⁺19, Section 6 and Appendix E]. $\square$

4 Zero-freeness for colour fields and Holant problems

4.1 Colour fields

4.1.1 Vertex-colour fields

We use the same letter Z for both partition functions: an argument z denotes the edge-activity polynomial, while an argument λ denotes the field polynomial defined below. For $G = (V, E) \in \mathcal{G}_\Delta$, a pinning $\tau : \Lambda \rightarrow [q]$, and field variables $\lambda = (\lambda_{u,c})_{(u,c) \in V \times [q]}$, define the normalized pinned field polynomial

$$\hat{Z}_G^\tau(\lambda) := \sum_{\varphi: V^\tau \rightarrow [q]} \prod_{u \in V^\tau} \lambda_{u, \varphi_u} \prod_{\substack{uv \in E \\ \{u,v\} \not\subseteq \Lambda}} \mathbf{1}_{[(\tau \cup \varphi)_u \neq (\tau \cup \varphi)_v]}. \quad (37)$$

Here $\mathbf{1}$ denotes the all-one vector on $V \times [q]$. At $\lambda = \mathbf{1}$, this is the hard-colouring value $\hat{Z}_G^\tau(0) > 0$ when $q \geq \Delta + 1$. The unpinned field polynomial is $Z_G(\lambda) = \hat{Z}_G^\emptyset(\lambda)$. The pinned field polynomial is

$$Z_G^\tau(\lambda) := \sum_{\substack{\sigma: V \rightarrow [q] \text{ proper} \\ \sigma|_\Lambda = \tau}} \prod_{v \in V} \lambda_{v, \sigma_v}.$$

If τ is proper on $G[\Lambda]$, then

$$Z_G^\tau(\lambda) = \left(\prod_{v \in \Lambda} \lambda_{v, \tau_v} \right) \hat{Z}_G^\tau(\lambda|_{V^\tau \times [q]}); \quad (38)$$

otherwise $Z_G^\tau(\lambda) \equiv 0$.

For $v \in V^\tau$, conditioning on its colour gives

$$\hat{Z}_G^\tau(\lambda) = \sum_{a: b_v^\tau(a)=0} \lambda_{v,a} \hat{Z}_G^{\tau^{v=a}}(\lambda|_{(V^\tau \setminus \{v\}) \times [q]}). \quad (39)$$

The normalized pinned function remains defined even for a blocked colour; only its coefficient in the parent recursion is zero.

Theorem 4.1 (Lee–Yang polydisc). *Let q, Δ be integers with $\Delta \geq 2$. Each of the following parameter assumptions yields the conclusion below:*

- (i) $q \geq (11/6 - 1/84000)\Delta$;
- (ii) $\Delta \geq 125$ and $q \geq 1.809\Delta$;
- (iii) $\Delta \geq 3$ and $q \geq \Delta + 3$.

In cases (i) and (ii), put $\mathcal{G} = \mathcal{G}_\Delta$. In case (iii), there is an integer $g_{LY} = g_{LY}(q, \Delta) \geq 3$ for which we put $\mathcal{G} = \mathcal{G}_{\Delta, g_{LY}}$. For each applicable case there is $\theta = \theta(q, \Delta) \in (0, 1/2]$ such that, for every $G \in \mathcal{G}$ and every pinning τ ,

$$\widehat{Z}_G^\tau(\lambda) \neq 0 \quad \text{whenever } |\lambda_{u,c} - 1| < \theta \quad (u \in V^\tau, c \in [q]).$$

For a proper pinning, the corresponding statement for $Z_G^\tau(\lambda)$ follows from (38) provided $\lambda_{v,\tau_v} \neq 0$ for every $v \in \Lambda$. In particular, it holds when the same polydisc condition is imposed at every vertex and colour.

Proof. The coupling-independence input at $x = 0$ is a coupling theorem for each regime, proved in the companion article (see Section A). In regime (i), the strict-Vigoda and Carlson–Vigoda bounds cover all cases except the critical pairs $(\Delta, q) = (6j, 11j)$, $1 \leq j \leq 20$, where the hard-colouring input is [CFG⁺25, Theorem 20]. Regime (ii) uses the Carlson–Vigoda bound, and regime (iii) uses the large-girth argument based on [CLMM23], which also supplies $g_{LY} = g_*(q, \Delta)$. In each regime the coupling theorem gives a finite coupling-independence constant $C_0 = C_0(q, \Delta)$ under every pinning in the stated induced-subgraph-closed graph class. These coupling theorems concern the normalized child laws in (8), including colours blocked at the root, and their proofs do not use the field result being proved here.

We use the separator induction of Lemma 3.2, with base field $\mathbf{1}$ and coupling constant C_0 . Choose L, s, N, α as in that proof. Induct on the number of free vertices, proving nonvanishing and the response bound

$$\left| \text{Log} \frac{\widehat{Z}_G^{\tau^{v=a}}(\lambda) / \widehat{Z}_G^{\tau^{v=a}}(\mathbf{1})}{\widehat{Z}_G^{\tau^{v=b}}(\lambda) / \widehat{Z}_G^{\tau^{v=b}}(\mathbf{1})} \right| \leq \alpha. \quad (40)$$

All normalized pinned base values are positive by Lemma 2.2, and logarithms continue from $\lambda = \mathbf{1}$. Bounded components are handled by continuity over finitely many local instances.

For a larger component, choose a shell S with $W_{1, \text{Ham}}(\nu_a, \nu_b) \leq C_0/L$. Include the inside and shell field weights in the local polynomial $D_{c,\xi}(\lambda)$, so that

$$\widehat{Z}_G^{\tau^{v=c}}(\lambda) = \sum_{\xi} D_{c,\xi}(\lambda) \widehat{Z}_{G'}^{\tau_\xi}(\lambda).$$

Field perturbations preserve the hard constraints. Thus each local polynomial is either identically zero or has positive value at $\mathbf{1}$. The finite family of nonzero local polynomials admits a common $\theta > 0$ such that

$$\eta_c(\xi) := \text{Log} \frac{D_{c,\xi}(\lambda)}{D_{c,\xi}(\mathbf{1})} \quad \text{satisfies} \quad |\eta_c(\xi)| \leq \alpha/8.$$

Set $\eta_c(\xi) = 0$ for identically zero local polynomials; those assignments have zero ν_c -mass. The exterior response

$$h_\xi = \text{Log} \frac{\widehat{Z}_{G'}^{\tau_\xi}(\lambda)}{\widehat{Z}_{G'}^{\tau_\xi}(\mathbf{1})}$$

is α -Lipschitz in Hamming distance by induction and has oscillation at most $s\alpha \leq 1/16$. The separator identity becomes

$$R_c := \frac{\widehat{Z}_G^{\tau^{v=c}}(\lambda)}{\widehat{Z}_G^{\tau^{v=c}}(\mathbf{1})} = \mathbb{E}_{\nu_c}[e^{h+\eta_c}].$$

Consequently Lemma 3.1 gives $|\text{Log}(R_a/R_b)| \leq 2\alpha C_0/L + \alpha/2 < \alpha$, proving (40).

Finally, take a colour a_0 allowed at v . After division by R_{a_0} , the parent recursion is

$$\frac{\hat{Z}_G^\tau(\lambda)}{R_{a_0}} = \sum_{a: b_v^\tau(a)=0} \hat{Z}_G^{\tau^{v=a}}(\mathbf{1}) \lambda_{v,a} \frac{R_a}{R_{a_0}}.$$

Shrink θ to at most $1/2$. Each summand then has positive real part, since its first factor is positive and its argument has absolute value at most $\pi/4 + \alpha < \pi/2$. This closes the induction. $\square$

4.1.2 Edge-colour fields

For a finite simple graph $G = (V, E)$, let $L(G)$ be its line graph, whose vertex set is E . Proper edge colourings of G are proper vertex colourings of $L(G)$. An edge-colour pinning is a map $\tau : \Lambda_E \rightarrow [q]$ for some $\Lambda_E \subseteq E$; it is proper when adjacent pinned edges have distinct colours. With field variables $\lambda = (\lambda_{e,c})_{e \in E, c \in [q]}$, define

$$\hat{Z}_G^{\text{edge}, \tau}(\lambda) := \hat{Z}_{L(G)}^\tau(\lambda), \quad Z_G^{\text{edge}, \tau}(\lambda) := Z_{L(G)}^\tau(\lambda). \quad (41)$$

Thus $Z_G^{\text{edge}} := Z_G^{\text{edge}, \emptyset}$. If τ is proper, then

$$Z_G^{\text{edge}, \tau}(\lambda) = \left(\prod_{e \in \text{dom } \tau} \lambda_{e, \tau_e} \right) \hat{Z}_G^{\text{edge}, \tau}(\lambda); \quad (42)$$

otherwise $Z_G^{\text{edge}, \tau}(\lambda) \equiv 0$.

Corollary 4.2 (Edge-colour Lee–Yang polydisc). *Let q, Δ be integers with $\Delta \geq 2$ and $q \geq 3\Delta$. There exists $\theta = \theta(q, \Delta) \in (0, 1/2]$ such that, for every finite simple graph $G = (V, E)$ of maximum degree at most Δ and every edge-colour pinning τ ,*

$$\hat{Z}_G^{\text{edge}, \tau}(\lambda) \neq 0 \quad \text{whenever} \quad |\lambda_{e,c} - 1| < \theta \quad (e \in E \setminus \text{dom } \tau, c \in [q]).$$

In particular, $Z_G^{\text{edge}}(\lambda) \neq 0$ in the same polydisc. For a proper pinning, $Z_G^{\text{edge}, \tau}(\lambda) \neq 0$ when the same condition is imposed at every edge-colour pair.

Proof. Put $\Delta_L = 2\Delta - 2$ and let $\mathcal{H}_\Delta$ be the class of line graphs $L(G)$ with $\Delta(G) \leq \Delta$. Every graph in $\mathcal{H}_\Delta$ has maximum degree at most Δ_L , and the class is closed under induced subgraphs.

At $x = 0$, the edge-Potts coupling theorem of the companion article (see Section A) gives the explicit Hamming coupling-independence constant $C_0 = \Delta - 1$ for all normalized edge-colour pinnings, on the common set of remaining free edges. This supplies the required hard-colouring input on $\mathcal{H}_\Delta$. Also $q \geq 3\Delta \geq \Delta_L + 1$, so all normalized hard base values are positive. Repeating the separator induction from the proof of Theorem 4.1, with degree bound Δ_L and coupling constant C_0 , gives the claimed polydisc. The fully pinned conclusion follows from (42). $\square$

4.2 Log-concave Boolean Holant problems

We next apply the method to log-concave Boolean Holant problems, using only the notation needed here. A symmetric Boolean signature of arity d is specified by

$$f : \{0, \dots, d\} \longrightarrow [0, \infty),$$

where $f(k)$ is the value assigned when exactly k incident edges are selected. Such a signature is *log-concave* if its positive support is an interval and

$$f(k)^2 \geq f(k-1)f(k+1) \quad (1 \leq k < d).$$

For a graph $G = (V, E)$, signatures $\mathbf{f} = (f_v)_{v \in V}$ with $\text{arity}(f_v) = \deg_G(v)$, and edge activities $\mathbf{z} = (z_e)_{e \in E}$, put

$$Z_{G,\mathbf{f}}(\mathbf{z}) := \sum_{S \subseteq E} \prod_{v \in V} f_v(\deg_S(v)) \prod_{e \in S} z_e, \quad (43)$$

where $\deg_S(v)$ counts the selected edges incident to v . Zero locations and approximation algorithms for Holant polynomials were studied in [GLLZ19, CFF⁺22]. For log-concave signatures satisfying $f_v(0) > 0$, Chen and Gu established spectral independence and rapid sampling through recursive couplings [CG24], while He, Li, Qiu, and Zhang obtained an FPTAS [HLQZ25].

4.2.1 Residual signatures and coupling independence

Lemma 4.3 (Residual Holant input). *Fix an integer $\Delta \geq 2$, a finite family $\mathcal{F}$ of log-concave symmetric Boolean signatures satisfying $f(0) > 0$ for every $f \in \mathcal{F}$, and $R > 0$. Let $\mathcal{F}_{\text{res}}$ consist of all normalized residual signatures obtained as follows: for every $f \in \mathcal{F}$ and integers $j, r \geq 0$ with $j + r \leq \text{arity}(f)$ and $f(j) > 0$, take*

$$g(k) := \frac{f(j+k)}{f(j)} \quad (0 \leq k \leq r)$$

in $\mathcal{F}_{\text{res}}$. This is a finite family of log-concave signatures, each with initial-interval support and $g(0) = 1$.

There is $C = C(\Delta, \mathcal{F}, R) < \infty$ with the following property. Let $H = (W, F)$ be a finite simple graph of maximum degree at most Δ , carrying at each vertex w a signature $g_w \in \mathcal{F}_{\text{res}}$ of arity $\deg_H(w)$, and let $\mathbf{x} \in [0, R]^F$. For an edge $e = uv \in F$, its normalized 0-child is obtained by deleting e and restricting the endpoint signatures to their first $\deg_{H-e}(w) + 1$ entries. If $g_u(1)g_v(1) > 0$, its normalized 1-child instead has endpoint signatures

$$k \mapsto \frac{g_w(k+1)}{g_w(1)} \quad (0 \leq k \leq \deg_{H-e}(w)), \quad w \in \{u, v\}.$$

All other signatures are unchanged, and every child signature belongs to $\mathcal{F}_{\text{res}}$. The normalization removes the fixed-edge factor $z_e g_u(1)g_v(1)$, so the 1-child remains defined when $x_e = 0$. Let $\mu_{\mathbf{x}}^{e=0}$ and $\mu_{\mathbf{x}}^{e=1}$ be their Gibbs laws on $\{0, 1\}^{F \setminus \{e\}}$. Then

$$W_{1,\text{Ham}}(\mu_{\mathbf{x}}^{e=0}, \mu_{\mathbf{x}}^{e=1}) \leq C.$$

If $Z^{e=0}$ and $Z^{e=1}$ denote the corresponding normalized child partition functions, then

$$Z^{e=1}(\mathbf{x}) \leq Z^{e=0}(\mathbf{x}). \quad (44)$$

When $g_u(1)g_v(1) = 0$, the unnormalized 1-child is identically zero.

Proof. The claims about $\mathcal{F}_{\text{res}}$ follow because restriction and multiplication by a positive scalar preserve log-concavity; the assumption $f(0) > 0$ makes every surviving support an initial interval. Let

$$A := \max(\{0\} \cup \{g(1) : g \in \mathcal{F}_{\text{res}}, \text{arity}(g) \geq 1\}).$$

When all free activities are positive, apply the two endpoint shifts successively, shifting each original sequence before restricting it to the remaining degree. An isolated endpoint changes only a removed scalar and requires no coupling. By [CG24, Proposition 17 and Lemma 21], applying the two endpoint comparisons as in the proof of Proposition 14 there gives

$$C = 2((1 + A^2 R)^\Delta - 1).$$

Here the intermediate shifted signatures remain in $\mathcal{F}_{\text{res}}$. If some activities vanish, replace only those coordinates by positive values tending to zero. Each normalized child has the all-zero configuration with weight one, so both Gibbs laws converge on the finite common state space. Compactness of the coupling polytope gives a limiting coupling with the same bound.

For the final assertion, the successive ratios of a log-concave signature g are nonincreasing on its positive support. Since $g(0) = 1$, multiplying these ratios gives

$$\frac{g(t+k)}{g(t)} \leq g(k) \quad (g(t) > 0, \quad t+k \leq \text{arity}(g)). \quad (45)$$

If the numerator vanishes, the inequality is immediate. Taking $t = 1$, each normalized 1-child endpoint signature is pointwise at most its 0-child counterpart. The termwise comparison over subsets of $F \setminus \{e\}$ proves (44). $\square$

4.2.2 Zero-freeness on the nonnegative orthant

Theorem 4.4 (Log-concave Holant polytube). *Fix an integer $\Delta \geq 2$ and a finite family $\mathcal{F}$ of log-concave symmetric Boolean signatures satisfying $f(0) > 0$ for every $f \in \mathcal{F}$. For every $R > 0$, there is $\varepsilon = \varepsilon(\Delta, \mathcal{F}, R) > 0$ such that the following holds. For every finite simple graph $G = (V, E)$ of maximum degree at most Δ , every assignment $(f_v)_{v \in V}$ with $f_v \in \mathcal{F}$ of arity $\deg_G(v)$, and every $\mathbf{z} \in \mathbb{C}^E$,*

$$Z_{G, \mathbf{f}}(\mathbf{z}) \neq 0 \quad \text{whenever} \quad \mathbf{z} \in \mathcal{U}_\varepsilon([0, R])^E.$$

The activities may be perturbed independently. In particular, $\bigcup_{R>0} \mathcal{U}_{\varepsilon(\Delta, \mathcal{F}, R)}([0, R])^E$ is an open zero-free neighborhood of $[0, \infty)^E$. Here R is common to all coordinates and is fixed before the width ε is chosen; the union is taken after the Cartesian power. On the diagonal $z_e = z$, the univariate specialization $Z_{G, \mathbf{f}}(z\mathbf{1})$ is nonzero throughout the open set

$$U_{\Delta, \mathcal{F}} := \bigcup_{R>0} \mathcal{U}_{\varepsilon(\Delta, \mathcal{F}, R)}([0, R]) \supset [0, \infty),$$

which depends only on $(\Delta, \mathcal{F})$ and is common to all the graphs and signature assignments above.

Proof. Let $\mathcal{F}_{\text{res}}$ be the residual family constructed in Lemma 4.3 for the present $\Delta, \mathcal{F}$. We prove the stronger assertion for every normalized instance on a graph of maximum degree at most Δ whose signatures belong to $\mathcal{F}_{\text{res}}$. This class is closed under structurally surviving pinnings, after removing the fixed activity factors and normalizing the remaining signatures at zero. Dividing each original f_v by $f_v(0) > 0$ puts the instance in this class and only multiplies its partition function by a positive constant.

The interaction graph of the edge variables has maximum degree at most $\Delta_L := 2(\Delta - 1)$. Let C be the constant from Lemma 4.3, take an integer $L \geq \max\{1, 64C\}$, and set

$$s := \Delta_L(\Delta_L - 1)^{L-1}, \quad N := 1 + \Delta_L \sum_{k=0}^{L-1} (\Delta_L - 1)^k.$$

Choose $0 < \alpha < \pi/4$ with $s\alpha \leq 1/16$. There are only finitely many connected normalized instances with at most N edges, up to their graph and signature data. Compactness over $[0, R]^N$ therefore gives a common local radius on which their partition functions are nonzero and all surviving child quotients below have logarithm at most α .

We now prove simultaneously, by strong induction on the number of free edges, that for every normalized instance $H = (W, F)$, its partition function is nonzero whenever $\max_{e' \in F} |z_{e'} - x_{e'}| < \varepsilon$, for $\mathbf{x} \in [0, R]^F$, and every edge $e \in F$ whose unnormalized 1-child is not identically zero satisfies

$$\left| \text{Log} \frac{Z^{e=1}(\mathbf{z})/Z^{e=1}(\mathbf{x})}{Z^{e=0}(\mathbf{z})/Z^{e=0}(\mathbf{x})} \right| \leq \alpha. \quad (46)$$

The logarithm is analytically continued from $\mathbf{z} = \mathbf{x}$. The assertion is immediate with no free edge.

Fix an instance $H = (W, F)$ with $n := |F| \geq 1$, and fix an edge $e \in F$ whose 1-child survives. We first prove the response bound. Components of the interaction graph not containing e give the same factors in the two children; the induction hypothesis makes them nonzero, so they cancel. We may therefore replace H by the component containing e , relabel it as $H = (W, F)$, reset $n := |F|$, and write Z for its partition function. In its interaction graph, let

$$S_j(e) := \{e' \in F : \text{dist}_{L(H)}(e, e') = j\}, \quad 1 \leq j \leq L.$$

If some $S_j(e)$ is empty, this component has at most N edges and the local compactness choice applies. Otherwise, project a coupling of the two real child laws onto these L disjoint shells. Their projected costs sum to at most C , so one shell $S = S_{j_*}(e)$ has cost at most C/L ; also $|S| \leq s$. Put

$$I := \{e' \in F \setminus \{e\} : 1 \leq \text{dist}_{L(H)}(e, e') < j_*\}, \quad O := \{e' \in F : \text{dist}_{L(H)}(e, e') > j_*\}.$$

For $i \in \{0, 1\}$, let $\Xi_i \subseteq \{0, 1\}^S$ be the shell assignments admitting a completion in the i -child whose signature product is nonzero, treating the activities as indeterminates. Initial-interval support makes each Ξ_i downward closed and ensures that it contains $\mathbf{0}$. The termwise comparison in Lemma 4.3 also gives $\Xi_1 \subseteq \Xi_0$; we use the common domain $\Xi := \Xi_0$.

To identify the exterior explicitly, let W_O be the vertices incident to O , and let $t_w(\xi)$ count selected shell edges at w . No vertex in W_O is incident to e or I , by the definition of the line-graph shells. For $\xi \in \Xi$, its normalized exterior signature is therefore

$$g_{w,\xi}(k) = \frac{g_w(t_w(\xi) + k)}{g_w(t_w(\xi))} \quad (w \in W_O, \quad 0 \leq k \leq \deg_O(w)).$$

Structural feasibility and initial-interval support make every denominator positive. These signatures belong to $\mathcal{F}_{\text{res}}$ and are independent of the root state. Write Z_ξ for their partition function on O . Sum over I , absorbing the shell monomial and the removed factors $g_w(t_w(\xi))$ into $D_{i,\xi}$, to obtain

$$Z^{e=i}(\mathbf{z}) = \sum_{\xi \in \Xi_i} D_{i,\xi}(\mathbf{z}) Z_\xi(\mathbf{z}), \quad i \in \{0, 1\}.$$

Structurally impossible assignments are absent from the sum; an assignment selecting an edge with zero base activity remains in the sum and merely has zero real weight.

By induction, $Z_\xi(\mathbf{z}) \neq 0$, so define

$$h_\xi := \text{Log} \frac{Z_\xi(\mathbf{z})}{Z_\xi(\mathbf{x})} \quad (\xi \in \Xi).$$

Write $h(\xi) := h_\xi$. If $\xi, \xi' \in \Xi$ differ in one coordinate, fix the root and every edge of I to zero, and all other shell edges to their common states. Leaving only the changed shell edge and the exterior free gives a residual instance with $|O| + 1 < n$ free edges. Both assignments remain structurally feasible when the inside is set to zero, by initial-interval support; its two normalized children are exactly $Z_\xi, Z_{\xi'}$. Both logarithm branches vanish at $\mathbf{z} = \mathbf{x}$, so uniqueness of analytic continuation identifies $h_\xi - h_{\xi'}$ with the inductive response logarithm. Any two elements of Ξ can be joined through $\xi \wedge \xi'$ by a Hamming path contained in Ξ . Hence

$$|h_\xi - h_{\xi'}| \leq \alpha \text{Ham}(\xi, \xi'), \quad \max_{\xi, \xi' \in \Xi} |h_\xi - h_{\xi'}| \leq s\alpha \leq \frac{1}{16}.$$

Extend the real shell marginals by zero from Ξ_i to Ξ :

$$\nu_i(\xi) := \frac{D_{i,\xi}(\mathbf{x})Z_\xi(\mathbf{x})}{Z^{e=i}(\mathbf{x})} \quad (\xi \in \Xi_i), \quad \nu_i(\xi) := 0 \quad (\xi \notin \Xi_i).$$

Their Hamming–Wasserstein distance is at most C/L . Put $M_i := \mathbb{E}_{\nu_i}[e^h]$. Adding and subtracting the local factors at $\mathbf{x}$ gives

$$\begin{aligned} \frac{Z^{e=i}(\mathbf{z})}{Z^{e=i}(\mathbf{x})} &= M_i + E_i, \\ E_i &= \frac{\sum_{\xi \in \Xi_i} (D_{i,\xi}(\mathbf{z}) - D_{i,\xi}(\mathbf{x})) Z_\xi(\mathbf{x}) e^{h_\xi}}{Z^{e=i}(\mathbf{x})}. \end{aligned}$$

Each $D_{i,\xi}$ involves at most $|I \cup S| \leq N - 1$ variables. Only vertices touching S contribute nontrivial exterior normalization constants, so there are at most $2s$ such factors. All local data thus range over a finite family. After requiring $\varepsilon \leq 1$, there is a graph-independent $K \geq 1$ such that

$$|D_{i,\xi}(\mathbf{z}) - D_{i,\xi}(\mathbf{x})| \leq K\varepsilon \quad (i \in \{0, 1\}, \xi \in \Xi_i).$$

The all-zero inside assignment gives $D_{i,0}(\mathbf{x}) \geq 1$, whence $Z^{e=i}(\mathbf{x}) \geq Z_0(\mathbf{x})$. Repeated pointwise use of (45) on the exterior gives $Z_\xi(\mathbf{x}) \leq Z_0(\mathbf{x})$. Enlarging K to absorb $|\Xi|$ and $e^{1/16}$, we obtain

$$|E_i| \leq K\varepsilon |e^{h_0}|, \quad \left| \frac{M_i}{e^{h_0}} - 1 \right| \leq e^{1/16} - 1, \quad |M_i| \geq \frac{3}{4} |e^{h_0}|.$$

These bounds distinguish structural feasibility from zero base activities and are therefore uniform on the closed box $[0, R]^F$.

Apply Lemma 3.1 on Ξ , with $A = \alpha$, $d = \text{Ham}$, and no local perturbation. It gives

$$\left| \text{Log} \frac{M_1}{M_0} \right| \leq \frac{2\alpha C}{L} \leq \frac{\alpha}{32}.$$

The preceding disk estimate fixes the same logarithm branch as continuation from $\mathbf{z} = \mathbf{x}$. For $u_i := E_i/M_i$, we have $|u_i| \leq 4K\varepsilon/3$. Require $\varepsilon < \min\{3/(8K), 3\alpha/(32K)\}$. Then $|u_i| < 1/2$, and $|\text{Log}(1 + u_i)| \leq 2|u_i|$ bounds the two corrections in

$$\text{Log} \frac{M_1 + E_1}{M_0 + E_0} = \text{Log} \frac{M_1}{M_0} + \text{Log}(1 + u_1) - \text{Log}(1 + u_0)$$

by a total of at most $16K\varepsilon/3 < \alpha/2$. This proves (46) with room to spare.

Having proved the response bound for every surviving child pair, choose any edge $e = uv$ to prove nonvanishing for the current instance, and put $a_e := g_u(1)g_v(1)$. If $a_e = 0$, the 1-child is identically zero and $Z(\mathbf{z}) = Z^{e=0}(\mathbf{z}) \neq 0$ by induction. Otherwise,

$$Z(\mathbf{z}) = Z^{e=0}(\mathbf{z}) + a_e z_e Z^{e=1}(\mathbf{z}).$$

The possible positive values of a_e form a finite set, and

$$r := \frac{Z^{e=1}(\mathbf{x})}{Z^{e=0}(\mathbf{x})} \leq 1$$

by (44). With ℓ denoting the logarithm in (46),

$$\frac{Z(\mathbf{z})}{Z^{e=0}(\mathbf{z})} = 1 + a_e x_e r e^\ell + a_e (z_e - x_e) r e^\ell.$$

The first two terms have real part at least one, while the last has modulus at most $a_e e^\alpha \varepsilon$. A final uniform reduction of ε makes this modulus smaller than one and closes the induction.

For a point in the stated polytube, choose $x_e \in [0, R]$ with $|z_e - x_e| < \varepsilon$ for every edge and apply the induction. Taking the union over $R > 0$ gives the asserted open neighborhood of $[0, \infty)^E$. On the diagonal, a single such R works for all coordinates, giving the asserted univariate domain. $\square$

4.2.3 b -matchings and b -edge covers

Corollary 4.5 (b -matchings). *Fix an integer $\Delta \geq 2$ and $R > 0$. There is $\varepsilon = \varepsilon(\Delta, R) > 0$ such that, for every finite simple graph $G = (V, E)$ of maximum degree at most Δ and every integer vector $\mathbf{b} = (b_v)_{v \in V}$ with $0 \leq b_v \leq \deg_G(v)$, and every $\mathbf{z} \in \mathbb{C}^E$,*

$$Z_{G, \mathbf{b}}^{\text{match}}(\mathbf{z}) := \sum_{\substack{S \subseteq E \\ \deg_S(v) \leq b_v \ \forall v}} \prod_{e \in S} z_e \neq 0$$

whenever $\mathbf{z} \in \mathcal{U}_\varepsilon([0, R])^E$. Thus $\bigcup_{R>0} \mathcal{U}_{\varepsilon(\Delta, R)}([0, R])^E$ is an open zero-free neighborhood of $[0, \infty)^E$.

Proof. Apply Theorem 4.4 to the finite family $f_{d,b}(k) = \mathbf{1}_{[k \leq b]}$ for integers $0 \leq b \leq d \leq \Delta$ and $0 \leq k \leq d$. $\square$

Corollary 4.6 (b -edge covers). *Fix an integer $\Delta \geq 2$ and $0 < \lambda_- \leq \lambda_+ < \infty$. There is $\varepsilon = \varepsilon(\Delta, \lambda_-, \lambda_+) > 0$ such that, for every finite simple graph $G = (V, E)$ of maximum degree at most Δ and every integer vector $\mathbf{b} = (b_v)_{v \in V}$ with $0 \leq b_v \leq \deg_G(v)$, and every $\mathbf{z} \in \mathbb{C}^E$,*

$$Z_{G, \mathbf{b}}^{\text{cover}}(\mathbf{z}) := \sum_{\substack{S \subseteq E \\ \deg_S(v) \geq b_v \ \forall v}} \prod_{e \in S} z_e \neq 0$$

whenever $\mathbf{z} \in \mathcal{U}_\varepsilon([\lambda_-, \lambda_+])^E$. Thus $\bigcup_{0 < \lambda_- \leq \lambda_+ < \infty} \mathcal{U}_{\varepsilon(\Delta, \lambda_-, \lambda_+)}([\lambda_-, \lambda_+])^E$ is an open zero-free neighborhood of $(0, \infty)^E$.

Proof. For $b'_v := \deg_G(v) - b_v$, complementation of the selected edges gives

$$Z_{G, \mathbf{b}}^{\text{cover}}(\mathbf{z}) = \left(\prod_{e \in E} z_e \right) Z_{G, \mathbf{b}'}^{\text{match}}((z_e^{-1})_{e \in E}).$$

Thus ∞ is the natural interpolation anchor, corresponding to $\mathbf{0}$ in the reciprocal variables. Inversion maps a sufficiently small neighborhood of the compact positive box $[\lambda_-, \lambda_+]^E$ into a polytube around $[\lambda_+^{-1}, \lambda_-^{-1}]^E$, so Corollary 4.5 applies. $\square$

The same diagonal specialization in Corollaries 4.5 and 4.6 gives common open univariate zero-free neighborhoods of $[0, \infty)$ and $(0, \infty)$, respectively, depending only on Δ .

A Further zero-free regimes for the Potts model

All zero-free theorems of this paper come from a single transfer principle, Theorem 2.4: on any induced-subgraph-closed class of graphs of maximum degree at most Δ with $q \geq \Delta + 1$, Hamming coupling independence at $x = 0$ and a uniform coupling bound on each $[\delta, 1]$, $\delta > 0$, yield a graph-uniform zero-free neighbourhood of $[0, 1]$ for all normalized pinned partition functions; the same induction gives the Lee–Yang polydiscs for vertex- and edge-colour fields. Every future coupling-independence theorem — for the Potts model in a new parameter range or on a new graph class, or for another local model with the same pinning and local-weight structure — therefore immediately yields, through the same framework, the corresponding zero-freeness and field-perturbation conclusions, with no further complex-analytic work.

We record the further regimes to which the framework has been applied. The coupling-independence proofs for these regimes were produced by AI models — GPT-5.6 Sol, except the girth-five results, which were proved by GPT-6 Astra — and appear in full in a companion article; all of them have been formally verified in Lean with GPT-6 Astra assistance, with the cited external results retained as explicit hypotheses [Shi26]. Table A.1 gives the additional zero-free regions. The coupling input of each row originates as follows: near-Vigoda, the strict flip coupling continued across the critical line, with [CFG⁺25, Theorem 20] at the critical pairs $(\Delta, q) = (6j, 11j)$; Carlson–Vigoda, the contraction of [CV25] extended to $x \in [0, 1]$; large-girth, the mechanism of [CLMM23] extended to $x \in [0, 1]$; high-temperature, Proposition 3.16; BBR, the certificate of [BBR25] at large girth; edge-Potts, a slot-lift coupling after [CWZZ25]; girth-five, a covariance recursion built on an all-pinning spectral gap proved in the companion article, converted to a coupling by the sphere-to-coupling theorem of [CLMM23, Lemma 5.13]; the girth-five setting parallels the spectral local-to-global framework of [CL26], from which no input is taken.

Regime	Parameters	Zero-free near
near-Vigoda	$\Delta \geq 2, q \geq (11/6 - 1/84000)\Delta$	$[0, 1]$
Carlson–Vigoda	$\Delta \geq 125, q \geq 1.809\Delta$	$[0, 1]$
large-girth	$\Delta \geq 3, q \geq \Delta + 3; \text{girth}(G^\tau) \geq g_*(q, \Delta)$	$[0, 1]$
high-temperature	$\Delta \geq 2, q > \frac{11}{6}(1 - x_*)\Delta, x_* \in (0, 1]$	$[x_*, 1]$
BBR interval	$q \geq 3, \Delta/q \geq (e - 1/2)/(e - 1); \text{girth}(G^\tau) \geq g_{\text{BBR}}(q, \Delta)$	$[x_0, 1]$
edge-Potts	$\Delta \geq 2, q \geq 3\Delta$	$[0, 1]$
girth-five	$q \geq (1 + \delta)\Delta, 0 < \delta \leq 1; \Delta \geq \Delta_5(\delta), \text{girth}(G^\tau) \geq 5$	$[0, 1]$

In every row the conclusion is $\widehat{Z}_G^\tau(z) \neq 0$ for all z in an ε -neighbourhood of the stated interval, with $\varepsilon > 0$ depending only on the listed parameters, uniformly over graphs and pinnings; the unnormalized pinned polynomial has no zeros in the same neighbourhood except possibly at $z = 0$, of order $m_G(\tau)$. In the edge-Potts row the polynomial is $\widehat{Z}_{L(G)}^\tau$, with τ an edge-colour pinning, and $m_G(\tau)$ is replaced by $m_{L(G)}(\tau)$. In the BBR row, $x_0 := 1 - \frac{q}{\Delta} \left(1 - \frac{k}{\Delta}\right)^2 \frac{\Delta-k}{\Delta-k/2}$ with $k := e^{\frac{\Delta-q/2}{\Delta-q}}$ (and $x_0 := 3/4$ if $(q, \Delta) = (3, 4)$). The girth-five row assumes no bounds on local short-cycle counts.

Table A.1: Additional zero-free regions obtained from the same framework.

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