Zero-Freeness is All You Need: A Weitz-Type FPTAS for the Entire Lee–Yang Zero-Free Region

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Abstract

We present a Weitz-type FPTAS for the ferromagnetic Ising model across the entire Lee–Yang zero-free region, without relying on the strong spatial mixing (SSM) property. Our algorithm is Weitz-type for two reasons. First, it expresses the partition function as a telescoping product of ratios, with the key being to approximate each ratio. Second, it uses Weitz's self-avoiding walk tree, and truncates it at logarithmic depth to give a good and efficient approximation. The key difference from the standard Weitz algorithm is that we approximate a carefully designed edge-deletion ratio instead of the marginal probability of a vertex's spin, ensuring our algorithm does not require SSM.

Furthermore, by establishing local dependence of coefficients (LDC), we indeed prove a novel form of SSM for these edge-deletion ratios, which, in turn, implies the standard SSM for the random cluster model. This is the first SSM result for the random cluster model on general graphs, beyond lattices. We prove LDC using a new division relation, and remarkably, such relations hold quite universally. As a result, we establish LDC for a variety of models. Combined with existing zero-freeness results for these models, we derive new SSM results for them. Our work suggests that both Weitz-type FPTASes and SSM can be derived from zero-freeness, while zero-freeness alone suffices for Weitz-type FPTASes, SSM additionally requires LDC, a combinatorial property independent of zero-freeness.

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1 Introduction

A *fully polynomial-time approximation scheme* (FPTAS) is a family of algorithms $\{A_\varepsilon\}$, where A_ε is a multiplicative $(1\pm\varepsilon)$-approximation algorithm with running time polynomial in $1/\varepsilon$ for each $\varepsilon > 0$. For counting problems, a standard approach to designing FPTASes is based on complex zero-free regions of the associated *partition function*. Once such a zero-free region is established, Barvinok's algorithm [Bar16] provides an FPTAS for approximating the partition function in a slightly smaller region. Specifically, suppose the partition function Z has no zeros in a complex region that contains a computationally tractable point. Then, possibly after a complex conformal mapping, the zero-freeness property ensures that $\log Z$ can be well-approximated in a slightly smaller region by its Taylor expansion series truncated at degree $O(\log n)$ where n is the instance size. More precisely, the Taylor expansion series f_k of $\log Z$ truncated at degree k gives an approximation of $\log Z$ within additive error Cr^{-k} for some positive constant C and $r > 1$. The coefficients of f_k can be computed via subgraph counting in time $\Delta^{O(k)}$ [PR17] where Δ is the maximum degree. Clearly, the running time is polynomial on n when $k = O(\log n)$. This approach connects the long-standing study of complex zeros of the partition function in statistical physics to algorithmic studies.

Another (and earlier) approach for devising FPTASes, originating in the work of Weitz [Wei06] and independently in Bandyopadhyay and Gamarnik [BG08], relies on the *correlation decay* property, or more precisely, *strong spatial mixing* (SSM). Roughly speaking, SSM asserts that correlations between spins decay exponentially with distance. Weitz's algorithm approximates the partition function defined on a graph G by expressing it as a telescoping product of certain marginal probabilities. The key technical ingredient of Weitz's algorithm is the *self-avoiding walk* (SAW) tree, which reduces the computation of a marginal probability on the original graph G to that on the SAW tree. However, the SAW tree may be exponentially large compared to the graph size n. The SSM property guarantees that the marginal probability can be well approximated by truncating the SAW tree at a depth of $O(\log n)$, making the evaluation efficient.

Both Barvinok's algorithm and Weitz's algorithm have been widely applied, especially to the study of 2-spin systems, which are among the most fundamental and well-studied models in statistical physics and counting problems. A 2-spin system is defined on a finite simple graph $G = (V, E)$ with parameters (β, γ, λ): two edge activities β, γ representing edge interactions, and a vertex activity λ representing an external field. A partial configuration of this system refers to a mapping $\sigma : \Lambda \to \{+, -\}$ for some $\Lambda \subseteq V$ which may be empty. When $\Lambda = V$, it is a configuration and is assigned a weight $w(\sigma) = \beta^{m_+(\sigma)}\gamma^{m_-(\sigma)}\lambda^{n_+(\sigma)}$, where $m_+(\sigma)$ and $m_-(\sigma)$ count $(+,+)$ and $(-,-)$ edges respectively, and $n_+(\sigma)$ counts vertices with spin $+$. The associated partition function is $Z_G(\beta, \gamma, \lambda) = \sum_{\sigma:V\to\{+,-\}} w(\sigma)$. Many natural combinatorial problems reduce to evaluating $Z_G(\beta, \gamma, \lambda)$. For instance, the case $(\beta = 0, \gamma = 1)$ corresponds to the hard-core model (independence polynomial), while $\beta = \gamma$ gives the celebrated Ising model. Depending on whether $\beta\gamma > 1$ or $\beta\gamma < 1$, the model is classified as ferromagnetic or antiferromagnetic, respectively.

Although FPTASes for 2-spin systems have been obtained via both Barvinok's algorithm [PR19, BCSV23, MB19, LSS19, PR20, GGHP22, PRS23, GLL20, SS21] and Weitz's algorithm [ZLB11, LLY12, LLY13, SST14], the applicability differs. While Barvinok's algorithm covers broad regions including ferromagnetic systems. Weitz's algorithm is mainly effective for antiferromagnetic systems where SSM holds. The SSM property crucially required by Weitz's algorithm is often absent in the ferromagnetic regime. Recent work [Reg23, SY24] established a connection between these two frameworks by showing that zero-freeness implies SSM, provided zero-free results hold for graphs with pinned vertices. As a consequence, some new SSM results have been proved to 2-

spin systems [SY24], which makes Weitz's algorithm can be applied. However, some of the most celebrated zero-freeness results, such as the Lee–Yang theorem [YL52, LY52] for the ferromagnetic Ising model, only hold for graphs without pinned vertices. Consequently, for the ferromagnetic Ising model on graphs of bounded degree, although Barvinok's algorithm yields an FPTAS throughout the Lee–Yang zero-free region [LSS19], i.e., $\lambda \in \mathbb{C}$ and $|\lambda| < 1$ or $|\lambda| > 1$ symmetrically, Weitz's algorithm cannot be applied to the entire zero-free region due to the lack of SSM. So far, the best known SSM results hold for regions much smaller than the Lee–Yang zero-free region [SY24, SS21], namely the union of the open disk centered at 0 with radius $\beta^{-\Delta}$ where Δ is the degree bound and a strip-shaped neighborhood of the real segment $[0, \frac{1}{\beta^{\Delta-2}((\Delta-2)\beta^2-\Delta)})$. In fact, it is known that SSM does not hold throughout the entire zero-free region [Bas07, SST14]. For instance, for the three-dimensional Ising model at low temperatures where the Lee–Yang theorem holds, it is known that SSM does not hold [Bas07], although weak spatial mixing does.

So far, for 2-spin systems, the regions where Barvinok's algorithm applies strictly contain and are much larger than those accessible to Weitz's algorithm. This raises a natural and interesting question: *Is Barvinok's algorithm inherently more powerful than Weitz's algorithm?* In this paper, we provide negative evidence for this question.

Our contributions

Theorem 1. *We present a Weitz-type FPTAS for the ferromagnetic Ising model throughout the entire Lee–Yang zero-free region, without requiring SSM.*

Our algorithm is a Weitz-type algorithm for two reasons. First, it expresses the partition function as a telescoping product of certain ratios and the key is to approximate each ratio. Secondly, in order to give a good approximation of the ratios, it uses the SAW tree and truncates it at logarithmic depth. However, crucial differences distinguish our algorithm from the standard Weitz algorithm, ensuring that our algorithm does not rely on SSM. First, instead of approximating the marginal probability $P_v = \frac{Z_{G,\sigma(v)=+}}{Z_G}$ of a vertex v being assign spin $+$, we approximate a carefully designed edge-deletion ratio $P_e = \frac{Z_{G-e}}{Z_G}$ where $G - e$ denotes the graph obtained from G by removing an edge e. Second, since SSM is unavailable, we cannot argue that truncating the SAW tree yields a good approximation. Inspired by Barvinok's method, we show that each edge-deletion ratio viewed as a function on λ can be well approximated by truncating its Taylor expansion series at logarithmic degree, and the coefficients can be computed efficiently via recursion on the SAW tree up to logarithmic depth.

The replacement of P_v by P_e is crucial for the above method to work. In fact, it is impossible to show that P_v could be approximated by logarithmic-degree Taylor truncation, since this would imply SSM throughout the Lee–Yang region contradicting known impossibility results [Bas07]. The reason is that, as shown in [SY24], the ratio P_v viewed as a function on λ and its SAW-tree version $P_v^{T_k}$ truncated at depth k share the same first k coefficients, known as *local dependence of coefficients* (LDC). Hence, if the Taylor expansion series f_k truncated at degree k approximates P_v well, i.e., $|P_v - f_k| \geq Cr^{-k}$ for some positive constants C and $r > 1$, then f_k also approximates $P_v^{T_k}$ well. Then, by a triangle inequality, one would have $|P_v - P_v^{T_k}| \leq 2Cr^k$, which is the standard SSM. This argument further implies that if LDC can be established for edge-deletion ratios, then together with zero-freeness, which guarantees good approximation by Taylor series of logarithmic degree, we obtain a form of SSM for these ratios. In other words, zero-freeness plus LDC implies SSM. In this paper, we further show that such a LDC property holds. Thus, we establish SSM for edge-deletion ratios[1] across the entire zero-free region.

Theorem 2 (SSM for edge-deletion ratios)**.** *Fix $\beta > 1$ and $\lambda \in \mathbb{D}$. Then there exist constants $C > 0$ and $r > 1$ such that for every graph $G = (V, E)$, for any edge $e \in E$ and subsets $A, B \subseteq E \setminus \{e\}$, we have*

$$|P_{G-A,e} - P_{G-B,e}| \ \leq \ Cr^{-\,d_G(e,\,A \neq B)},$$

where $P_{G,e}$ denotes the edge-deletion ratio Z_{G-e}/Z_G and $d_G(e, A \neq B)$ is the distance in G between the edge e and the set of edges on which the boundary conditions differ.

Surprisingly, the choice of the edge-deletion ratio is not only crucial for a Weitz-type FPTAS for the Ising model, but it also admits a probabilistic interpretation in the Ising related random cluster model. It corresponds exactly to the marginal probability that an edge is included in the random cluster model. Thus, our edge-deletion SSM implies standard SSM for the random cluster model. This is the first SSM result for the random cluster model on general graphs, whereas all previous results were confined to lattices.

The LDC property was first implicitly shown for the hard-core model [Reg23] via cluster expansion, and later formally introduced and generalized to 2-spin systems [SY24] using a Christoffel–Darboux type identity. In this paper, we extend this framework by proving that an explicit identity is unnecessary: a more general division relation, established via a delicate one-to-one mapping, suffices. Quite remarkably, such relations hold quite universally. Consequently, we establish LDC for diverse models, including the Potts model, the hypergraph independence polynomial, and the Holant framework, even in regions where zero-freeness fails. Thus, LDC is revealed as a combinatorial property independent of zero-freeness. Together with known zero-freeness results for the above models, we obtain new SSM results.

In summary, this paper suggests the following relationship between zero-freeness, SSM, Weitz-type FPTASes, and LDC. Both Weitz-type FPTASes and SSM can be derived from zero-freeness, but with different requirements:

- For Weitz-type FPTASes, zero-freeness alone suffices.
- For SSM, one additionally needs LDC, a combinatorial property independent of zero-freeness.

Organization

The paper is organized as follows. In Section 3, to derive the Weitz-type algorithm, we show that the truncated series provides a good approximation of the edge-deletion ratio from zero-freeness via complex-analytic tools, and then analyze the truncated series through the self-avoiding walk tree and operations on formal power series. This yields the FPTAS. In Section 4, we introduce the framework in which zero-freeness implies SSM via LDC, and establish the SSM property in terms of edge activities for the ferromagnetic Ising model, using the Lee–Yang theorem and the divisibility relation. This edge-based SSM property further implies the standard SSM of the corresponding random-cluster model. In Section 5, we extend the divisibility relation to various models and derive their SSM properties.

[1] We actually prove a more general form of SSM; see Theorem 13.

2 Preliminaries

2.1 Ising model

For a graph $G = (V, E)$, with edge activities $\boldsymbol{\beta} = (\beta_e)_{e \in E}$ and vertex activities $\boldsymbol{\lambda} = (\lambda_v)_{v \in V}$, the weight of a configuration $\sigma : V \to \{+, -\}$ is given by

$$w(\sigma) = \prod_{e \in m(\sigma)} \beta_e \prod_{v \in n(\sigma)} \lambda_v$$

where $m(\sigma) = \{e = (u, v) \in E \mid \sigma_u = \sigma_v\}$ is the set of edges whose endpoints have the same spin, and $n(\sigma) = \{v \in V \mid \sigma_v = +\}$ is the set of vertices assigned spin $+$. The patition function of the Ising model is defined by $Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \sum_{\sigma : V \to \{+,-\}} w(\sigma)$. The celebrated Lee–Yang theorem states the zero-free region of the Ising model.

Theorem 3 (Lee–Yang theorem). *Let $G = (V, E)$ be a graph with parameters $\boldsymbol{\beta} \in [1, \infty)^E$ and $\boldsymbol{\lambda} \in \mathbb{D}^V$ where $\mathbb{D}$ is the unit open disk in the complex plane. Then the partition function of Ising model $Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda}) \neq 0$.*

A partial configuration of the Ising model is a mapping $\sigma : \Lambda \to \{+, -\}$ for some $\Lambda \subseteq V$, which my be empty. The conditional partition function is defined as $Z_G^{\sigma_\Lambda}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \sum_{\substack{\sigma : V \to \{+,-\} \\ \sigma|_\Lambda = \sigma_\Lambda}}$ where $\sigma|_\Lambda$ denotes the restriction of the configuration σ on Λ. Let $u, v \in V$, then define

$$Z_{G,v}^{\sigma_\Lambda,+}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \sum_{\substack{\sigma : V \to \{+,-\} \\ \sigma|_\Lambda = \sigma_\Lambda, \sigma(v) = +}} w(\sigma), \quad \text{and} \quad Z_{G,v,u}^{\sigma_\Lambda,+,+}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \sum_{\substack{\sigma : V \to \{+,-\} \\ \sigma|_\Lambda = \sigma_\Lambda, \sigma(v) = \sigma(u) = +}} w(\sigma).$$

Then the conditional marginal probability that v is pinned to $+$ given the partial configuration σ_Λ and the corresponding marginal ratio are defined as

$$P_{G,v}^{\sigma_\Lambda}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \frac{Z_{G,v}^{\sigma_\Lambda,+}(\boldsymbol{\beta}, \boldsymbol{\lambda})}{Z_G^{\sigma_\Lambda}(\boldsymbol{\beta}, \boldsymbol{\lambda})}, \quad \text{and} \quad R_{G,v}^{\sigma_\Lambda}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \frac{Z_{G,v}^{\sigma_\Lambda,+}(\boldsymbol{\beta}, \boldsymbol{\lambda})}{Z_{G,v}^{\sigma_\Lambda,-}(\boldsymbol{\beta}, \boldsymbol{\lambda})}.$$

2.2 Weitz's tree reduction

In the seminal work of Weitz [Wei06], the self-avoiding walk (SAW) tree reduces the computation of marginal probabilities for 2-spin models on general graphs to the corresponding computation on trees. We do not repeat the construction of the SAW tree here, and refer readers to [Wei06] for details. We only state the key property of the SAW tree. If the graph $G = (V, E)$ has maximum degree Δ, then the SAW tree $T_{\text{SAW}}(G, v)$ rooted at $v \in V$ also has maximum degree Δ. Moreover, the marginal probability and marginal ratio at v in G coincide with those at the root of $T_{\text{SAW}}(G, v)$ (with some leaves possibly pinned).

Consider a rooted tree $T = (V, E)$ with root $r \in V$. Suppose r has d children $v_1, \ldots, v_d$. Removing r yields d subtrees $T_1, \ldots, T_d$, where T_i is rooted at v_i. Let R_{T_i,v_i} denote the marginal ratio at v_i in T_i (e.g., $R_{T_i,v_i} = Z_{T_i,v_i}^+ / Z_{T_i,v_i}^-$), and let $R_{T,r}$ be the marginal ratio at r in T. Then the tree recursion expressing $R_{T,r}$ in terms of the R_{T_i,v_i} is given by a multivariate map $F_d : \hat{\mathbb{C}}^d \to \hat{\mathbb{C}}$:

$$R_{T,r} \;=\; F_d(R_{T_1,v_1}, \ldots, R_{T_d,v_d}), \qquad F_d(x_1, \ldots, x_d) \;=\; \lambda \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma},$$

where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, β, γ are the edge-interaction parameters, and λ is the external field. For the ferromagnetic Ising model, we have $\beta = \gamma > 1$. If the root v_i of T_i is pinned to $+$ or $-$, then we set $R_{T_i,v_i} = \infty$ or 0, and the term $(\beta R_{T_i,v_i} + 1)/(R_{T_i,v_i} + \gamma)$ is interpreted as β or $1/\gamma$.

Weitz's algorithm [Wei06] approximates the partition function of the 2-spin system on a graph G by a telescoping product of marginal probabilities. Then the SAW tree reduction reduces the problem to approximating marginal probabilities on trees. The strong spatial mixing property on trees guarantees that the marginal probability at the root can be approximated by truncating the tree at logarithmic depth. The standard strong spatial mixing of the Ising model is given below.

Definition 4 (Strong spatial mixing)**.** *Fix parameters β, λ and a family of graphs $\mathcal{G}$. The Ising model defined on $\mathcal{G}$ with parameters (β, λ) is said to satisfy strong spatial mixing with exponential rate $r > 1$ if there exists a constant C such that for any $G = (V, E) \in \mathcal{G}$, any vertices $v \in V$, any partial configuration σ_{Λ_1} and τ_{Λ_2}, we have*

$$\left| P_{G,v}^{\sigma_{\Lambda_1}}(\beta, \lambda) - P_{G,v}^{\tau_{\Lambda_2}}(\beta, \lambda) \right| \le C r^{-d_G(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}$$

where $\sigma_{\Lambda_1} \neq \tau_{\Lambda_2}$ denotes the set $(\Lambda_1 \backslash \Lambda_2) \cup (\Lambda_2 \backslash \Lambda_1) \cup \{v \in \Lambda_1 \cap \Lambda_2 : \sigma_{\Lambda_1}(v) \neq \tau_{\Lambda_2}(v)\}$, i.e., the set of vertices where σ_{Λ_1} and τ_{Λ_2} differ. The term $d_G(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})$ denotes the shortest path distance from v to any vertex in $\sigma_{\Lambda_1} \neq \tau_{\Lambda_2}$.

However, the best known SSM results for the ferromagnetic Ising model with $\beta > 1$ apply only to graphs of bounded degree Δ, and they hold in a region much smaller than the Lee–Yang zero-free region [SY24, SS21], namely the union of the open disk centered at 0 with radius $\beta^{-\Delta}$ and a neighborhood of the real segment $\left[0, \dfrac{1}{\beta^{\Delta-2}\left((\Delta-2)\beta^2 - \Delta\right)}\right)$.

Consequently, Weitz's algorithm cannot be applied to obtain an FPTAS for the ferromagnetic Ising model over the entire Lee–Yang zero-free region. Nonetheless, by analyzing the edge deletion ratios and leveraging zero-freeness via truncated Taylor expansions together with the SAW-tree reduction, we establish a Weitz-type FPTAS that works throughout the full Lee–Yang zero-free region.

2.3 Complex analysis tools and truncated power series

A *region* is a connected open set in $\mathbb{C}$. In particular, an open disk with one interior point removed is also a region. We denote by $\mathbb{D}_\rho(z_0)$ the open disk centered at z_0 with radius ρ, and by $\partial\mathbb{D}_\rho(z_0)$ its boundary circle. If $z_0 = 0$, we simply write $\mathbb{D}_\rho$ and $\partial\mathbb{D}_\rho$; if $\rho = 1$, we omit the subscript ρ. Let $f(z) = \sum_{k \ge 0} a_k z^k$ be a (formal) power series, and write its truncation to degree m as

$$f^{[m]} := \sum_{k=0}^{m} a_k z^k \quad (= f \bmod z^{m+1}).$$

The following lemma is a standard result of complex analysis textbook, derived from Cauchy's integral formula.

Lemma 5. *Let $f(z)$ be analytic in a neighborhood of $z = 0$, and suppose $|f(z)| \le M$ on the circle $\partial\mathbb{D}_\rho$ for some $\rho > 0$. Then for any $z \in \mathbb{D}_\rho$, we have*

$$|f(z) - f^{[n]}(z)| \;\le\; \frac{M}{\rho(r-1)r^n}, \qquad \text{where } r = \rho/|\lambda| > 1.$$

Applying Lemma 5 requires a uniform bound on a circle for a family of analytic functions. In [Reg23], Regts employed Montel's theorem to obtain such bounds, leading to the following result.

Lemma 6. *Let $\mathbb{U}$ be a region, and let $\mathcal{F}$ be a family of holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ such that $f(\mathbb{U}) \subseteq \mathbb{C} \setminus \{0, 1\}$ for all $f \in \mathcal{F}$. If there exist a point $z_0 \in \mathbb{U}$ and a constant C such that $|f(z_0)| \leq C$ for all $f \in \mathcal{F}$, then for any compact subset $S \subseteq \mathbb{U}$, there exists another constant C' such that for all $f \in \mathcal{F}$ and all $z \in S$, we have $|f(z)| \leq C'$.*

Next, assume the first $m+1$ coefficients of f and g are given. We measure running time in terms of arithmetic operations over $\mathbb{C}$. Using FFT-based polynomial multiplication (see [VZGG03]), the following bounds hold:

1. Scalar multiplication: $(kf)^{[m]} = k\, f^{[m]}$ in $O(m)$ time.
2. Addition: $(f + g)^{[m]} = f^{[m]} + g^{[m]}$ in $O(m)$ time.
3. Multiplication: $(fg)^{[m]}$ in $O(m \log m)$ time (via FFT).
4. Division: if $g(0) \neq 0$, then $(f/g)^{[m]}$ in $O(m \log m)$ time by Newton iteration.

In particular, each of the above truncated operations can be done in $O(m \log m)$ time with FFT.

3 Weitz-type algorithm for the ferromagnetic Ising model

Our approach is a telescoping algorithm based on edge deletion. For a graph $G = (V, E)$ with parameters (β, λ), we order the edges in E as $e_1, e_2, \ldots, e_m$, denote $G_i = (V, E_i)$ where $E_i = \{e_1, e_2, \ldots, e_i\}$ for $1 \leq i \leq m$ and $G_0 = (V, \varnothing)$. Then we have

$$\frac{1}{Z_G(\lambda)} = \prod_{i=1}^{m} \frac{Z_{G_{i-1}}(\lambda)}{Z_{G_i}(\lambda)} \frac{1}{Z_{G_0}(\lambda)} = (1 + \lambda)^{-|V|} \prod_{i=1}^{m} P_{G_i, e_i}(\lambda),$$

where we define the edge–deletion ratio $P_{G,e} = Z_{G-e}/Z_G$. Thus, approximating $Z_G(\lambda)$ within a multiplicative factor of ε reduces to approximating each ratio $P_{G_i,e_i}(\lambda)$ within a multiplicative error of at most $\varepsilon/(4m)$, for all $1 \leq i \leq m$. This is achieved by computing the truncated Taylor series of $P_{G_i,e_i}(\lambda)$ at $\lambda = 0$ up to degree $k = O(\log(m/\varepsilon))$, and then evaluating it at λ.

3.1 Truncated series is a good approximation

To show the truncation series gives a good approximation via Lemma 5, we apply Lemma 6 to obtain a uniform bound of $P_{G,e}(\lambda)$ for all graph G and edge $e \in G$. This requires that $P_{G,e}(\lambda)$ avoid 0 and 1 for all graph G and edge $e \in G$.

Lemma 7. *Let $G = (V, E)$ be a graph. For edge activity $\beta > 1$ and external field $\lambda \in \mathbb{D}$, the edge-deletion ratio $P_{G,e}(\beta, \lambda)$ omits the values 0 and 1.*

Proof. See Lemma 22 as a special case. The result follows entirely from the Lee–Yang zero-free region. □

Then a uniform bound of $P_{G,e}(\lambda)$ for all graph G and edge $e \in G$ follows from Lemma 6. Where the upper bound (for a circle) is used to establish the additive error in Lemma 5, and the lower bound (for a single point) is used to turn the additive error into a multiplicative error.

Lemma 8. *Fix $\beta > 1$ and $S \subseteq \mathbb{D}$ be a compact set. There exists a constants $M, b > 0$ such that $b \le |P_{G,e}(\lambda)| \le M$ for all graph G, edge $e \in G$ and $\lambda \in S$.*

Proof. Fix $\beta > 1$ and a compact $S \subseteq \mathbb{D}$. Let $\mathcal{F} = \{P_{G,e}(z) : G \text{ a graph}, e \in E(G)\}$. By Lemma 7, every $f \in \mathcal{F}$ omits $\{0, 1\}$ on $\mathbb{D}$, and $f(0) = 1/\beta$ is uniformly bounded. Hence, by Lemma 6, there exists $M > 0$ such that $|P_{G,e}(z)| \le M$ for all $z \in S$, all G, and all e.

For the lower bound, apply the same argument to $\mathcal{F}^{-1} = \{1/P_{G,e}(z)\}$. Each $g \in \mathcal{F}^{-1}$ also omits $\{0, 1\}$ and $g(0) = \beta$ is uniformly bounded, so there exists $M' > 0$ with $|1/P_{G,e}(z)| \le M'$ for all $z \in S$. Setting $b := 1/M'$ yields $b \le |P_{G,e}(z)| \le M$ for all $z \in S$, as claimed. □

Lemma 9. *Fix $\beta > 1$ and $\lambda \in \mathbb{D}$. Then there exists $k = O(\log(m/\varepsilon))$ such that for every graph $G = (V, E)$ with $m = |E|$ and every edge $e \in E$, the truncated series $P_{G,e}^{[k]}$ evaluated at λ, satisfies the relative bound*

$$\frac{\left|P_{G,e}(\lambda) - P_{G,e}^{[k]}(\lambda)\right|}{\left|P_{G,e}(\lambda)\right|} \le \frac{\varepsilon}{4m}.$$

Proof. Pick $\rho = \frac{1+|\lambda|}{2}$ so that $\lambda \in \mathbb{D}_\rho$. By Lemma 8, there exists $M > 0$ such that $|P_{G,e}(z)| \le M$ for all $z \in \partial\mathbb{D}_\rho$ and all G, e. Applying Lemma 5 to $P_{G,e}$ yields

$$\left|P_{G,e}(\lambda) - P_{G,e}^{[k]}(\lambda)\right| \le C\, r^{-k}, \qquad r = \rho/|\lambda| > 1,$$

for some constant C independent of G, e. Moreover, Lemma 8 with $S = \{\lambda\}$ provides a uniform lower bound $b > 0$ such that $|P_{G,e}(\lambda)| \ge b$ for all G, e. Therefore,

$$\frac{\left|P_{G,e}(\lambda) - P_{G,e}^{[k]}(\lambda)\right|}{\left|P_{G,e}(\lambda)\right|} \le \frac{C}{b}\, r^{-k}.$$

Choosing $k = \left\lceil \frac{\log\left((4mC)/(\varepsilon b)\right)}{\log r} \right\rceil$ guarantees the desired bound. Since C, b and r are independent of G, e, this choice satisfies $k = O(\log(m/\varepsilon))$ uniformly over all G, e. □

Weitz's algorithm computes Z_G as a telescoping product of conditional marginal probabilities. We choose to truncate the edge-deletion ratios rather than the marginals for the following reasons. First, if the truncated series yields a uniformly exponential error bound for the marginal probabilities, as in [SY24], this would imply strong spatial mixing. However, the best-known SSM region for the ferromagnetic Ising model (even on bounded-degree graphs) is much smaller than the Lee–Yang zero-free region, and SSM is known not to hold throughout that region [Bas07, SST14]. Second, with pinning, the zero-free region for the partition function is strictly smaller than the Lee–Yang zero-free region, so the marginal probabilities (being ratios involving pinned partition functions) need not be well defined or holomorphic on the entire Lee–Yang zero-free region.

3.2 Computing the truncated series via Weitz's tree reduction

For simplicity, we omit the parameters (β,λ) in the following. Suppose $e_i=(u,v)$. By the definition of the Ising model, we have

$$
\begin{aligned}
P_{G_i,e_i}=\frac{Z_{G_{i-1}}}{Z_{G_i}}=&\frac{Z^{+,+}_{G_{i-1},u,v}+Z^{-,-}_{G_{i-1},u,v}+Z^{-,+}_{G_{i-1},u,v}+Z^{+,-}_{G_{i-1},u,v}}{Z^{+,+}_{G_i,u,v}+Z^{-,-}_{G_i,u,v}+Z^{-,+}_{G_i,u,v}+Z^{+,-}_{G_i,u,v}}\\
=&\frac{\frac{1}{\beta}Z^{+,+}_{G_i,u,v}+\frac{1}{\beta}Z^{-,-}_{G_i,u,v}+Z^{-,+}_{G_i,u,v}+Z^{+,-}_{G_i,u,v}}{Z^{+,+}_{G_i,u,v}+Z^{-,-}_{G_i,u,v}+Z^{-,+}_{G_i,u,v}+Z^{+,-}_{G_i,u,v}}\\
=&1+\left(1-\frac{1}{\beta}\right)\frac{Z^{+,+}_{G_i,u,v}+Z^{-,-}_{G_i,u,v}}{Z^{+,+}_{G_i,u,v}+Z^{-,-}_{G_i,u,v}+Z^{-,+}_{G_i,u,v}+Z^{+,-}_{G_i,u,v}}\\
=&1+\left(1-\frac{1}{\beta}\right)\frac{R^{u^+}_{G_i,v}R^{v^-}_{G_i,u}+1}{R^{u^+}_{G_i,v}R^{v^-}_{G_i,u}+1+R^{u^-}_{G_i,v}+R^{v^-}_{G_i,u}}.
\end{aligned}
$$

The second line holds because if u and v have the same spin, the only extra contribution in Z_{G_i} (compared with $Z_{G_{i-1}}$) is the edge $e_i=(u,v)$, which contributes a factor of β. The last line follows by dividing numerator and denominator by $Z^{-,-}_{G_i,u,v}$ and substituting the ratio identities

$$
\frac{Z^{+,+}_{G_i,u,v}}{Z^{-,-}_{G_i,u,v}}=\frac{Z^{+,+}_{G_i,u,v}}{Z^{+,-}_{G_i,u,v}}\cdot\frac{Z^{+,-}_{G_i,u,v}}{Z^{-,-}_{G_i,u,v}}=R^{u^+}_{G_i,v}R^{v^-}_{G_i,u},\qquad \frac{Z^{+,-}_{G_i,u,v}}{Z^{-,-}_{G_i,u,v}}=R^{v^-}_{G_i,u},\qquad \frac{Z^{-,+}_{G_i,u,v}}{Z^{-,-}_{G_i,u,v}}=R^{u^-}_{G_i,v}.
$$

To calculate $P_{G_i,e_i}(\lambda)^{[k]}$, it suffices to calculate the ratios $R^{u^+}_{G_i,v}(\lambda)^{[k]}$, $R^{v^-}_{G_i,u}(\lambda)^{[k]}$ and $R^{v^-}_{G_i,u}(\lambda)^{[k]}$. This can be done by the tree recursion on the self-avoiding walk tree $T_{\mathrm{SAW}}(G_i^{u^+},v)$, $T_{\mathrm{SAW}}(G_i^{v^-},u)$ and $T_{\mathrm{SAW}}(G_i^{u^-},v)$ respectively. Recall the tree recursion formula

$$
R_{T,r}(\lambda)=\lambda\prod_{i=1}^{d}\frac{\beta\, R_{T_i,v_i}(\lambda)+1}{R_{T_i,v_i}(\lambda)+\beta}.
$$

To compute $R_{T,r}(\lambda)^{[k]}$, it suffices to compute the truncated series of $\prod_{i=1}^{d}\frac{\beta\, R_{T_i,v_i}(\lambda)+1}{R_{T_i,v_i}(\lambda)+\beta}$ up to degree $k-1$; hence, for each child v_i we require $R_{T_i,v_i}(\lambda)^{[k-1]}$. Therefore $R_{T,r}(\lambda)^{[k]}$ can be obtained by traversing the truncated self-avoiding walk tree to depth k and, for every node at depth $i\in[0,k]$, computing the truncated series of its ratio to degree $k-i$. Suppose T has maximum degree Δ, at each node, multiplying at most Δ series and performing one division with FFT-based series arithmetic costs $O(\Delta\, k\log k)$. The truncated SAW tree to depth k has $O(\Delta^k)$ nodes. Hence the total running time is $O(\Delta^k\cdot\Delta\, k\log k)=O(\Delta^{k+1}k\log k)$.

If G has maximum degree Δ, then the self-avoiding walk tree $T_{\mathrm{SAW}}(G_i^{u^+},v)$, $T_{\mathrm{SAW}}(G_i^{v^-},u)$ and $T_{\mathrm{SAW}}(G_i^{u^-},v)$ also have maximum degree Δ. Thus, the time complexity for computing $P_{G_i,e_i}(\lambda)^{[k]}$ is $O(k\log k\Delta^{k+1})$.

3.3 Approximation and running time

Theorem 10. *Fix $\beta>1$ and $\lambda\in\mathbb{D}$. There exists a deterministic algorithm that, given a graph $G=(V,E)$ with $m=|E|$ and maximum degree Δ, and an accuracy parameter $\varepsilon\in(0,1)$, computes* *an approximation $\hat{Z}$ in time $\left(\frac{m}{\varepsilon}\right)^{O(\log\Delta)}$ such that*

$$\left|\frac{Z_G(\beta,\lambda)-\hat{Z}}{Z_G(\beta,\lambda)}\right| \le \varepsilon.$$

Proof. Choose $k = O(\log(m/\varepsilon))$ as in Lemma 9 and set $\hat{Z} = (1+\lambda)^{|V|}/\prod_{i=1}^m P^{[k]}_{G_i,e_i}(\lambda)$. For each i, let $\delta_i = \frac{P_{G_i,e_i}(\lambda)}{P^{[k]}_{G_i,e_i}(\lambda)} - 1$, thus $|\delta_i| = \left|\frac{P^{[k]}_{G_i,e_i}(\lambda)/P_{G_i,e_i}(\lambda)-1}{P^{[k]}_{G_i,e_i}(\lambda)/P_{G_i,e_i}(\lambda)}\right| \le \frac{\frac{\varepsilon}{4m}}{1-\frac{\varepsilon}{4m}} \le \frac{\varepsilon}{2m}$. Then we have

$$\left|\frac{Z_G(\beta,\lambda)-\hat{Z}}{Z_G(\beta,\lambda)}\right| = \left|1-\prod_{i=1}^m(1+\delta_i)\right| \le \exp\left(\sum_{i=1}^m|\delta_i|\right)-1 \le e^{\varepsilon/2}-1 \;\le\; \varepsilon.$$

By the SAW-tree recursion and FFT-based truncated series arithmetic, each $P^{[k]}_{G_i,e_i}(\lambda)$ is computable in $O(k\log k\,\Delta^{k+1})$ time; over all m edges the total time is

$$O(mk\log k\,\Delta^{k+1}) \;=\; \left(\frac{m}{\varepsilon}\right)^{O(\log\Delta)}. \qquad \square$$

Our algorithm shows that SSM is unnecessary for a Weitz-type FPTAS, as we do not rely on SSM for the marginal probability of a vertex being assigned a particular spin. Instead, it is crucial for our algorithm to replace marginal probabilities of vertices by edge-deletion ratios, which eliminates the need for SSM. In the next section, we show that, even though the standard notion of SSM does not hold, by further proving the local dependence of coefficients (LDC), a combinatorial property independent of zero-freeness, we can indeed establish a new form of SSM for edge deletion ratios. Thus, zero-freeness alone gives Weitz-type FPTASes, while zero-freeness plus LDC gives new forms of SSM.

4 SSM for Generalized Edge Ratios

SSM typically refers to the property that differences in conditional marginal probabilities at a given vertex exhibit exponential decay with respect to the distance of the disagreement condition in the Gibbs distribution.

If we ignore the probabilistic meaning of $P_{G,v}$, arithmetically, it is just a ratio of two partition functions conditioning on different partial configurations. Such a ratio can be extended to a much more general setting. For a partition function $Z_G(\boldsymbol{\beta},\boldsymbol{\lambda})$ viewed as a multivariate function on edge activities $(\beta_e)_{e\in E}$ and vertex external fields $(\lambda_v)_{v\in V}$, and a partial evaluation $m(V',E') : (\beta_e)_{e\in E'} \to [1,\infty), (\lambda_v)_{v\in V'} \to \mathbb{D}$ (i.e., substituting specific values for variables $(\beta_e)_{e\in E'}$ and $(\lambda_v)_{v\in V'}$), we consider the function

$$Z_G^{m(V',E')}((\beta_e)_{e\in E\setminus E'},(\lambda_v)_{v\in V\setminus V'})$$

where the values of $(\lambda_v)_{v\in V'}$ and $(\beta_e)_{e\in E'}$ are assigned by $m(V',E')$. When context is clear, we may omit the subscript $e\in E\setminus E'$ and $v\in V\setminus V'$ in $Z_G^{m(V',E')}$. Some particular partial evaluations have special meanings. For example, the assignment $m(u) : \lambda_u \to 0$ that assigns the external field λ_u of a particular vertex $u\in V$ to 0 gives the function $Z_G^{m(u)} = Z^-_{G,u}$ which is the partition function of the Ising model on the graph G with a pinned vertex u to the $-$ spin. Also, the assignment $m(e) : \beta_e \to 1$ that assigns the edge activity β_e of a particular edge $e\in E$ to 1 gives the function

$Z_G^{m(e)} = Z_{G-e}$ which is the partition function of the Ising model on the graph $G-e$, i.e., the graph obtained from G by removing the edge e.

In this paper, we focus on the partial evaluation $m(\emptyset, E')$ that only assigns values to edge activities for edges in E'. For simplicity, we write $m(\emptyset, E')$ as $m(E')$. Then, as an extension of the marginal probability $P_{G,v}$, we can define the ratio $P_{G,m(E')}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = Z_G^{m(E')}(\boldsymbol{\beta}, \boldsymbol{\lambda})/Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda})$ for any partial evaluation $m(E')$. Moreover, we can define the ratio conditioning on a pre-specified partial evaluation $m_1(E_1)$ by $P_{G,m(E')}^{m_1(E_1)}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = Z_G^{m_1(E_1),m(E')}(\boldsymbol{\beta}, \boldsymbol{\lambda})/Z_G^{m_1(E_1)}(\boldsymbol{\beta}, \boldsymbol{\lambda})$ for partial evaluation $m(E')$ satisfying $E' \cap E_1 = \emptyset$. If context is clear, we may omit the arguments $(\boldsymbol{\beta}, \boldsymbol{\lambda})$ and the specification of edge sets E_1 and E', and write $P_{G,m(E')}^{m_1(E_1)}(\boldsymbol{\beta}, \boldsymbol{\lambda})$ as $P_{G,m}^{m_1}$ for simplicity.

With these notations in hand, we are able to define the generalized form of edge-type SSM.

Definition 11 (Generalized edge-SSM)**.** *Let $\mathcal{G}$ be a family of graphs with parameters $(\boldsymbol{\beta}, \boldsymbol{\lambda})$ and $C_2 \geq C_1 \geq 0$ be constants. The Ising model defined on $\mathcal{G}$ is said to satisfy generalized edge-type strong spatial mixing (GE-SSM) with exponential rate $r > 1$ if there exists a constant C such that for any $G = (V, E) \in \mathcal{G}$, any $e \in E$ with $m = \{\beta_e \to \beta'_e\}$ where $\frac{\beta'_e}{\beta_e} \in [C_1, C_2]$ and any partial evaluation m_1, m_2 defined on $A, B \subseteq E \backslash \{v\}$ respectively, then*

$$\left| P_{G,m}^{m_1} - P_{G,m}^{m_2} \right| \leq C r^{-d_G(e, m_1 \neq m_2)}.$$

Here, we denote $m_1 \neq m_2 = (A \backslash B) \cup (B \backslash A) \cup \{f \in A \cap B : m_1(f) \neq m_2(f)\}$, which is the set of edges where m_1 and m_2 differ. The quantity $d_G(e, m_1 \neq m_2)$ is the shortest distance from any endpoint of e to any endpoint of an edge in $m_1 \neq m_2$.

If we restrict the partial evaluations $m(E')$ and $m_1(E_1)$ to assigning edge activities only to the value 1, then $P_{G,m(E')}^{m_1(E_1)} = \frac{Z_G^{m(E'),m_1(E_1)}}{Z_G^{m_1(E_1)}} = \frac{Z_{G-E'-E_1}}{Z_{G-E_1}}$. We define $P_{G,e} = \frac{Z_{G-e}}{Z_G}$. Then, as a special form of GE-SSM, we define the following edge-deletion form of SSM.

Definition 12 (Edge-deletion SSM)**.** *Let $\mathcal{G}$ be a family of graphs with parameters $(\boldsymbol{\beta}, \boldsymbol{\lambda})$. The Ising model defined on $\mathcal{G}$ is said to satisfy edge-deletion SSM with exponential rate $r > 1$ if there exists a constant C such that for any $G = (V, E) \in \mathcal{G}$, edge $e \in E$, sets of edge $A, B \subseteq E \backslash e$, then*

$$|P_{G-A,e} - P_{G-B,e}| \leq C r^{-d_G(e, A \neq B)}.$$

Indeed, we establish the GE-SSM result for the Ising model, as stated below.

Theorem 13. *Fix constants $\delta \in (0, 1)$ and $C_2 \geq C_1 \geq 0$. Then there exist constants $C > 0$ and $r > 1$ such that for all graph $G = (V, E)$ with parameters $\boldsymbol{\beta} \in [1, \infty)^E$ and $\boldsymbol{\lambda} \in (1-\delta)\mathbb{D}^V$, and for any edge $e \in E$ and sets $A, B \subseteq E \backslash \{e\}$, the following holds. Define the partial evaluation:*

$$m = \{\beta_e \to \beta'_e\}, \quad m_1 = \{\beta_f \to \beta_f^A\}_{f \in A}, \quad m_2 = \{\beta_f \to \beta_f^B\}_{f \in B}$$

where $\beta'_e \in [1, \infty)$, $\beta_f^A \in [1, \infty]$ for all $f \in A$, $\beta_f^B \in [1, \infty]$ for all $f \in B$ and $\frac{\beta'_e}{\beta_e} \in [C_1, C_2]$, we have

$$\left| P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda}) - P_{G,m}^{m_2}(\boldsymbol{\beta}, \boldsymbol{\lambda}) \right| \leq C r^{-d_G(e, m_1 \neq m_2)}.$$

Remark 14. *This theorem differs slightly from the definition of GE-SSM, as we allow the conditional partial evaluations to take the value ∞. However, this can be well-defined by taking limits of the corresponding ratios. Indeed, Lee–Yang theorem ensure the ratios $P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda})$ and $P_{G,m}^{m_2}(\boldsymbol{\beta}, \boldsymbol{\lambda})$*

is well-defined when for $\beta_f^A \in [1,\infty)$ for $f \in A$ and $\beta_f^B \in [1,\infty)$ for $f \in B$. Once the theorem is established in this setting, one can take the appropriate limits to extend its validity even when β_f^A or β_f^B approaches ∞.

If set $\beta'_e = 1$, then $\frac{\beta'_e}{\beta_e} \in [0,1]$ always holds. As a corollary, the edge-deletion SSM holds.

Corollary 15. *Fix $\delta \in (0,1)$. For any graph $G = (V,E)$ with $\boldsymbol{\beta} \in [1,\infty)^E$ and $\boldsymbol{\lambda} \in (1-\delta)\mathbb{D}^V$, the edge-deletion SSM holds.*

Such an edge-type SSM does not have an explicit probabilistic meaning in the Ising model. However, through the relationship between the Ising model and the random cluster model, we found that it can be interpreted as the standard SSM in the random cluster model.

4.1 LDC framework

For two complex functions $f(z)$ and $g(z)$ analytic near z_0, we denote by $(z-z_0)^k \mid f(z)-g(z)$ the property that their Taylor series expansions,

$$f(z) = \sum_{i=0}^{\infty} a_i (z-z_0)^i \quad \text{and} \quad g(z) = \sum_{i=0}^{\infty} b_i (z-z_0)^i$$

satisfy $a_i = b_i$ for $0 \le i \le k-1$.

The following lemma is a key tool in establishing SSM from zero-freeness, as used in [Reg23, SY24]. It also follows as a consequence of Lemma 5.

Lemma 16. *Let $f(z)$ and $g(z)$ be two analytic functions on some complex neighborhood U of z_0. Suppose that the $(z-z_0)^n \mid f(z)-g(z)$. Also, suppose that there exists an $M > 0$ such that both $|P(z)| \le M$ and $|Q(z)| \le M$ on some circle $\partial\mathbb{D}_\rho(z_0) \subseteq U$ ($\rho > 0$). Then for every $z \in \mathbb{D}_\rho(z_0)$, we have*

$$|f(z)-g(z)| \le \frac{2M}{\rho(r-1)r^{n-1}}, \quad \textit{with} \quad r = \frac{\rho}{|z-z_0|}.$$

In [SY24], Shao and Ye introduce the concept of local dependence of coefficients (LDC), which is implicitly used in [Reg23]. To establish edge-type SSM for the Ising model, we introduce LDC below.

Definition 17 (LDC)**.** *We say that the Ising model satisfies LDC if for all graphs $G = (V,E)$ with parameters $\boldsymbol{\beta} \in [1,\infty)^E$ and $\lambda \in \mathbb{D}$, the following holds. For an edge $e \in E$ and subsets $A, B \subseteq E \backslash \{e\}$, define the partial evaluations:*

$$m = \{\beta_e \to \beta'_e\}, \quad m_1 = \{\beta_f \to \beta_f^A\}_{f\in A}, \quad m_2 = \{\beta_f \to \beta_f^B\}_{f \in B}$$

where the modified parameters satisfy $\beta'_e \in [1,\infty)$, $\beta_f^A \in [1,\infty)$ for $f \in A$ and $\beta_f^B \in [1,\infty)$ for $f \in B$. It holds that

$$\lambda^{d_G(e, m_1 \neq m_2)+1} \mid P_{G,m}^{m_1}(\boldsymbol{\beta},\lambda) - P_{G,m}^{m_2}(\boldsymbol{\beta},\lambda).$$

To address the non-uniform external field, we prove a slightly modified form of LDC. Once we have the LDC and a uniform bound, we can establish the edge SSM.

Definition 18 (LDC)**.** *We say that the Ising model satisfies LDC if for all graphs $G = (V, E)$ with parameters $\boldsymbol{\beta} \in [1, \infty)^E$ and $\boldsymbol{\lambda} \in \mathbb{D}^V$, the following holds. For an edge $e \in E$ and subsets $A, B \subseteq E \backslash \{e\}$, define the partial evaluations:*

$$m = \{\beta_e \to \beta'_e\}, \quad m_1 = \{\beta_f \to \beta_f^A\}_{f \in A}, \quad m_2 = \{\beta_f \to \beta_f^B\}_{f \in B}$$

where the modified parameters satisfy $\beta'_e \in [1, \infty)$, $\beta_f^A \in [1, \infty)$ for $f \in A$ and $\beta_f^B \in [1, \infty)$ for $f \in B$. It holds that

$$z^{d_G(e, m_1 \neq m_2)+1} \mid P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - P_{G,m}^{m_2}(\boldsymbol{\beta}, \boldsymbol{\lambda} z).$$

4.2 Divisibility Relation via a Combinatorial Bijection

We establish a divisibility relation that implies the LDC.

Lemma 19. *Let $G = (V, E)$ be a graph with parameters $(\boldsymbol{\beta}, \boldsymbol{\lambda})$ where $\boldsymbol{\beta} \in [1, \infty)^E$ and $\boldsymbol{\lambda} \in \mathbb{D}^V$, Let $A, B \subseteq E$ be two disjoint edge sets, define the partial evaluations:*

$$m_1 = \{\beta_f \to \beta_f^A\}_{f \in A}, \quad m_2 = \{\beta_f \to \beta_f^B\}_{f \in B}$$

where the modified parameters satisfy $\beta_f^A \in [1, \infty)$ for $f \in A$ and $\beta_f^B \in [1, \infty)$ for $f \in B$. Then

$$z^{d_G(A,B)+1} \mid Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda} z) Z_G^{m_1, m_2}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - Z_G^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) Z_G^{m_2}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$$

where $d_G(A, B) = \min_{e_1 \in A, e_2 \in B} d_G(e_1, e_2)$.

Proof. For simplicity, we omit $(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$ in the notation. Let $\mathcal{S} = V \to \{+, -\}$, then

$$\begin{aligned} & Z_G Z_G^{m_1, m_2} - Z_G^{m_1} Z_G^{m_2} \\ = & \sum_{\sigma \in \mathcal{S}} w_G(\sigma) \sum_{\sigma \in \mathcal{S}} w_G^{m_1, m_2}(\sigma) - \sum_{\sigma \in \mathcal{S}} w_G^{m_1}(\sigma) \sum_{\sigma \in \mathcal{S}} w_G^{m_2}(\sigma) \\ = & \sum_{\substack{(\sigma_1, \sigma_2) \in \\ (\mathcal{S} \times \mathcal{S})}} w_G(\sigma_1) w_G^{m_1, m_2}(\sigma_2) - \sum_{\substack{(\sigma_3, \sigma_4) \in \\ (\mathcal{S} \times \mathcal{S})}} w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4) \end{aligned}$$

Let $R = \{(\sigma, \tau) \in \mathcal{S} \times \mathcal{S} : n_+(\sigma) + n_+(\tau) < d(A, B) + 1\}$, where $n_+(\sigma)$ is the number of vertices with $+$ spin in σ. We will show that there exists an automorphism f on R such that if $(\sigma_3, \sigma_4) = f(\sigma_1, \sigma_2)$, then $w_G(\sigma_1) w_G^{m_1, m_2}(\sigma_2) = w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4)$.

Let $(\sigma_1, \sigma_2) \in R$, consider the subgraph $H = (V, E_+(\sigma_1 | \sigma_2))$, where $\sigma_1 \mid \sigma_2$ denotes the logical OR, interpreting $+$ as true. Since $n_+(\sigma_1) + n_+(\sigma_2) < d(A, B) + 1$, there are no paths connecting any edge between A and B in H. Let S be the minimal vertex set containing all connected components of H that intersect with G_1 and $T = V \backslash S$. Swap the part at T of σ_1 and σ_2, write it as $(\sigma_3, \sigma_4) = (\sigma_1|_S \cup \sigma_2|_T, \sigma_2|_S \cup \sigma_1|_T)$. Obviously, $(\sigma_3, \sigma_4) \in R$ and the process is reversible (note $\sigma_3 | \sigma_4 = \sigma_1 | \sigma_2$, which is unchanged in the process), thus f is an automorphism.

Since there are no $(+, +)$ edges between S and T for $\sigma_1 | \sigma_2 = \sigma_3 | \sigma_4$, it follows that there are no $(+, +)$ edges between S and T for $\sigma_1, \sigma_2, \sigma_3$ and σ_4. For an edge $e = (u, v) \in E$ between S and T, define $s(e, \sigma) = \mathbb{1}[e \in E_-(\sigma)]$. Recalling that e cannot be a $(+, +)$ edge in any $\sigma_i (i = 1, 2, 3, 4)$, we obtain $s(e, \sigma) = 1 - \mathbb{1}[\sigma(u) = +] - \mathbb{1}[\sigma(v) = +]$. Moreover, note that $\mathbb{1}[\sigma_1(u) = +] + \mathbb{1}[\sigma_2(u) = +] = \mathbb{1}[\sigma_3(u) = +] + \mathbb{1}[\sigma_2(u) = +]$ and similarly for v. It follows that $s(e, \sigma_1) + s(e, \sigma_2) = s(e, \sigma_3) + s(e, \sigma_4)$.

Let $C = \{(u,v) \in E \mid u \in S, v \in T\}$ be the set of cut edges between S and T. By the definition of $w(\cdot)$, we have

$$
\begin{aligned}
& w_G(\sigma_1) w_G^{m_1,m_2}(\sigma_2) \\
=& \prod_{e\in C} \beta_e^{s(e,\sigma_1)} w_{G[S]}(\sigma_1|_S) w_{G[T]}(\sigma_1|_T) \prod_{e\in C} \beta_e^{s(e,\sigma_2)} w_{G[S]}^{m_1}(\sigma_2|_S) w_{G[T]}^{m_2}(\sigma_2|_T) \\
=& \prod_{e\in C} \beta_e^{s(e,\sigma_1)+s(e,\sigma_2)} w_{G[S]}^{m_1}(\sigma_2|_S) w_{G[T]}(\sigma_1|_T) w_{G[S]}(\sigma_1|_S) w_{G[T]}^{m_2}(\sigma_2|_T) \\
=& \prod_{e\in C} \beta_e^{s(e,\sigma_3)+s(e,\sigma_4)} w_{G[S]}^{m_1}(\sigma_3|_S) w_{G[T]}(\sigma_3|_T) w_{G[S]}(\sigma_4|_S) w_{G[T]}^{m_2}(\sigma_4|_T) \\
=& \prod_{e\in C} \beta_e^{s(e,\sigma_3)} w_{G[S]}^{m_1}(\sigma_3|_S) w_{G[T]}(\sigma_3|_T) \prod_{e\in C} \beta_e^{s(e,\sigma_4)} w_{G[S]}(\sigma_4|_S) w_{G[T]}^{m_2}(\sigma_4|_T) \\
=& w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4).
\end{aligned}
$$

Thus, the proof is complete. $\square$

4.3 Generalized edge-SSM

4.3.1 Edge-type LDC

Lemma 20. *Let $G = (V,E)$ be a graph with parameters $\boldsymbol{\beta} \in [1,\infty)^E$ and $\boldsymbol{\lambda} \in \mathbb{D}^V$. Let $e \in E$ and $A \subseteq E\backslash\{e\}$, $m = \{\beta_e \to \beta'_e\}$ with $\beta'_e \geq 1$ and $m_1 = \{\beta_f \to \beta'_f \mid f \in A\}$ where $\beta'_f \geq 1$ for all $f \in A$. Then the Taylor series of $P_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$ and $P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$ near $z = 0$ satisfy*

$$z^{d_G(e,A)+1} \mid P_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$$

Proof.

$$
\begin{aligned}
P_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) =& \frac{Z_G^m(\boldsymbol{\beta}, \boldsymbol{\lambda} z)}{Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda} z)} - \frac{Z_G^{m,m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)}{Z_G^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)} \\
=& \frac{Z_G^m(\boldsymbol{\beta}, \boldsymbol{\lambda} z) Z_G^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - Z_G^{m,m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda} z)}{Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda} z) Z_G^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)}.
\end{aligned}
$$

Clearly by Lee–Yang theorem, $\frac{1}{Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda} z) Z_G^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)}$ is analytic near $z = 0$. Combining this with Lemma 19, we have

$$z^{d_G(e,A)+1} \mid P_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z).$$

$\square$

Lemma 21 (LDC)**.** *Let $G = (V,E)$ be a graph with parameters $\boldsymbol{\beta} \in [1,\infty)^E$ and $\boldsymbol{\lambda} \in \mathbb{D}^V$. Let $e \in E$ and $A, B \subseteq E\backslash\{e\}$, and partial evaluations*

$$m = \{\beta_e \to \beta'_e\}, \quad m_1 = \{\beta_f \to \beta_f^A\}_{f\in A}, \quad m_2 = \{\beta_f \to \beta_f^B\}_{f\in B}$$

where $\beta'_e \in [1,\infty)$, $\beta_f^A \in [1,\infty)$ for $f \in A$ and $\beta_f^B \in [1,\infty)$ for $f \in B$. Then the Taylor series of $P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$ and $P_{G,m}^{m_2}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$ near $z = 0$ satisfy

$$z^{d_G(e,m_1 \neq m_2)+1} \mid P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - P_{G,m}^{m_2}(\boldsymbol{\beta}, \boldsymbol{\lambda} z).$$

Proof. Define $\boldsymbol{\beta}'$ as $\boldsymbol{\beta}$ after applying by $m_1 \cap m_2$, let $m_1' = m_1 \backslash m_2$ and $m_2' = m_2 \backslash m_1$, then

$$\begin{aligned}P^{m_1}_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) - P^{m_2}_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda} z) =& P^{m_1'}_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z) - P^{m_2'}_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z) \\ =& [P^{m_1'}_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z) - P_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z)] + [P_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z) - P^{m_2'}_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z)].\end{aligned}$$

By the previous lemma, we have $z^{d_G(e,m_1')+1} \mid P^{m_1'}_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z) - P_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z)$ and $z^{d_G(e,m_2')+1} \mid P_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z) - P^{m_2'}_{G,m}(\boldsymbol{\beta}', \boldsymbol{\lambda} z)$. Since $d_G(e, m_1 \neq m_2) = \min\{d_G(e, m_1'), d_G(e, m_2')\}$, we are done. $\square$

4.3.2 Uniform bound of edge-type ratio

We are ready to prove the edge type ratio avoid 0 and 1.

Lemma 22. *Let $G = (V, E)$ be a graph, with parameters $\boldsymbol{\beta} \in [1, \infty)^E$ and $\boldsymbol{\lambda} \in \mathbb{D}^V$, edge $e \in E$, if $\beta_e' \geq 1$ and $\beta_e' \neq \beta_e$, then $P_{G,\{\beta_e \to \beta_e'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda})$ avoid 0 and 1.*

Proof. Since $\beta_e' \geq 1$, by Lee–Yang theorem, it is trivial that $P_{G,\{\beta_e \to \beta_e'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda}) \neq 0$. We prove the ratio avoids 1.

Let $e = (u, v)$, we have

$$\begin{aligned}&Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda}) - Z_G^{\{\beta_e \to \beta_e'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda}) \\ =& Z^{+,+}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) + Z^{-,-}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) + Z^{+,-}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) + Z^{-,+}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) \\ &- \frac{\beta_e'}{\beta_e} Z^{+,+}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) - \frac{\beta_e'}{\beta_e} Z^{-,-}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) - Z^{+,-}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) - Z^{-,+}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) \\ =& \frac{\beta_e - \beta_e'}{\beta_e}(Z^{+,+}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) + Z^{-,-}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda})).\end{aligned}$$

Merge u, v into a single vertex w we get graph $G' = (V', E')$, set $\lambda_w = \lambda_u \lambda_v$, if parallel edges exist (i.e. $(u, x) \in E, (v, x) \in E$ for some $x \in V$), we merge them into a single edge and set $\beta_{(w,x)} = \beta_{(u,x)}\beta_{(v,x)}$. Write the partition function of G' with new parameters as $Z_{G'}(\boldsymbol{\beta}', \boldsymbol{\lambda}')$.

One can see $Z_{G'}(\boldsymbol{\beta}', \boldsymbol{\lambda}') = Z^{+}_{G',w}(\boldsymbol{\beta}', \boldsymbol{\lambda}') + Z^{-}_{G',w}(\boldsymbol{\beta}', \boldsymbol{\lambda}') = (Z^{+,+}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}) + Z^{-,-}_{G,u,v}(\boldsymbol{\beta}, \boldsymbol{\lambda}))/\beta_e$. Since $\boldsymbol{\lambda}' \in \mathbb{D}^{V'}$ and $\boldsymbol{\beta}' \in [1, \infty)^{E'}$, by Lee–Yang theorem, $Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda}) - Z_G^{\{\beta_e \to \beta_e'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = (\beta_e - \beta_e')Z_{G'}(\boldsymbol{\beta}', \boldsymbol{\lambda}') \neq 0$. Thus the ratio avoids 1. $\square$

Lemma 23 (uniform bound)**.** *Fix constant number $\delta \in (0, 1)$ and $C_2 \geq C_1 \geq 0$. Let S be a compact set of $\frac{1}{1-\delta}\mathbb{D}$. Then, there exists a constant $C > 0$ such that for any graph $G = (V, E)$ with parameters $\boldsymbol{\beta} \in [1, \infty)^E$ and $\boldsymbol{\lambda} \in (1-\delta)\mathbb{D}^V$, for any $e \in E$, any $\beta_e' \geq 1$ with $\frac{\beta_e'}{\beta_e} \in [C_1, C_2]$, we have $|P_{G,\{\beta_e \to \beta_e'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)| \leq C$ for all $z \in S$.*

Proof. Consider the family of functions $f(z) = P_{G,\{\beta_e \to \beta'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$ where z is the variant. It's trivial when $\beta_e' = \beta_e$, the ratio is exactly 1. So we only consider the family of ratio functions when $\beta_e' \neq \beta_e$. By Lemma 22, $P_{G,\{\beta_e \to \beta_e'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda} z)$ avoid 0 and 1 for all $z \in \frac{1}{1-\delta}\mathbb{D}$. Since $P_{G,\{\beta_e \to \beta_e'\}}(\boldsymbol{\beta}, \boldsymbol{\lambda} \cdot 0) = \frac{\beta_e'}{\beta_e} \in [C_1, C_2]$ is bounded, by Lemma 6, the upper bound is got. $\square$

Now we are ready to prove edge-type SSM of the Ising model and then immediately deduce the SSM of the random cluster model.

Proof of Theorem 13. By Lemma 21, we have $z^{d_G(e,m_1\neq m_2)+1} \mid P^{m_1}_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda}z) - P^{m_2}_{G,m}(\boldsymbol{\beta}, \boldsymbol{\lambda}z)$. Let $S = (1+\delta)\partial\mathbb{D}$, which is a compact subset of $\frac{1}{1-\delta}\mathbb{D}$. By Lemma 23 we know that the ratio is uniformly bounded for $z \in S$. Choosing $z = 1 \in (1+\delta)\mathbb{D}$, we apply Lemma 16 to conclude the proof. $\square$

4.4 SSM for random cluster model

Let $G = (V, E)$ be a graph, $\boldsymbol{p} \in [0,1]^E$, $\boldsymbol{\lambda} \in [0,1]^V$ be parameters. The weight of a configuration $S \subseteq E$ in the (weighted) random cluster model is defined by:

$$w^{\mathrm{RC}}_{G,\boldsymbol{p},\boldsymbol{\lambda}}(S) = \prod_{e\in S} p_e \prod_{e\in E\setminus S}(1-p_e) \prod_{C\in\kappa(V,S)} \left(1 + \prod_{j\in C}\lambda_j\right),$$

where $\kappa(V, S)$ denotes the set of connected components of graph (V, S). The partition function of the random cluster model is given by

$$Z^{\mathrm{RC}}_G(\boldsymbol{p}, \boldsymbol{\lambda}) = \sum_{S\subseteq E} w^{\mathrm{RC}}_{G,\boldsymbol{p},\boldsymbol{\lambda}}(S).$$

When $\boldsymbol{\lambda} = \mathbf{1}$, the weighted random cluster model reduces to the standard random cluster model for the Ising model without external field. The relationship between the Ising model with an external field and the random cluster model is given in the following lemma.

Lemma 24 ([FGW23, Proposition 2.1])**.** *Let $G = (V,E)$ be a graph, and let $\boldsymbol{\beta} \in [1, +\infty)^E$ and $\boldsymbol{\lambda} \in [0,1]^V$ be parameters. Then,*

$$Z^{\mathrm{Ising}}_G(\boldsymbol{\beta}, \boldsymbol{\lambda}) = \left(\prod_{e\in E}\beta_e\right) Z^{\mathrm{RC}}_G(\boldsymbol{p}, \boldsymbol{\lambda}),$$

where $\boldsymbol{p} = 1 - \boldsymbol{\beta}^{-1} = (1-\beta_e^{-1})_{e\in E}$.

Remark 25. *Expressing it as $Z^{\mathrm{RC}}_G(\boldsymbol{p}, \boldsymbol{\lambda}) = Z^{\mathrm{Ising}}_G(\boldsymbol{\beta}, \boldsymbol{\lambda})/\prod_{e\in E}\beta_e$, we observe that setting $\beta_e = \infty$ is well-defined by taking the limit, which corresponds to setting $p_e = 1$ in the random cluster model.*

When $\boldsymbol{p} \in [0,1]^E$ and $\boldsymbol{\lambda} \in [0,1]^V$, RC model induces a distribution $\mu(\cdot)$ where $\mu(S) = w(S)/Z$ for $S \subseteq E$. Denote the marginal probability on an edge e such that e is picked and unpicked as $P^+_{G,e}(\boldsymbol{p}, \boldsymbol{\lambda}) = Z^+_{G,e}/Z_G$ and $P^-_{G,e}(\boldsymbol{p}, \boldsymbol{\lambda}) = Z^-_{G,e}/Z_G$ where $Z^+_{G,e} = \sum_{S\subseteq E, e\in S} w(S)$ and $Z^-_{G,e} = \sum_{S\subseteq E\setminus e} w(S)$ respectively. We also define the partition function conditioning on a pre-described partial configuration σ_A ($A \subseteq E$, each edge in A is pinned to be in or out the configurations, we use the notation $+$ and $-$ denoting in and out) denoted by

$$Z^{\sigma_A}_G = \sum_{\substack{S\subseteq E\\ S|_A=\sigma_A}} w^{\mathrm{RC}}_{G,\boldsymbol{p},\boldsymbol{\lambda}}(S)$$

and then the conditional marginal probabilities e unpicked under condition σ_A are defined by

$$P^{\sigma_A}_{G,e} = \frac{Z^{\sigma_A,-}_{G,e}}{Z^{\sigma_A}_G}.$$

Definition 26 (SSM for the random cluster model)**.** *Let $\mathcal{G}$ be a family of graphs with parameters $(\boldsymbol{p}, \boldsymbol{\lambda})$. The random cluster model defined on $\mathcal{G}$ is said to satisfy strong spatial mixing with exponential rate $r > 1$ if there exists a constant C such that for any $G = (V, E) \in \mathcal{G}$, any edge $e \in V$, any partial configuration σ_{Λ_1} and τ_{Λ_2} where $\Lambda_1, \Lambda_2 \subseteq E \backslash e$, we have*

$$\left| P_{G,e}^{\sigma_{\Lambda_1}}(\boldsymbol{p}, \boldsymbol{\lambda}) - P_{G,e}^{\tau_{\Lambda_2}}(\boldsymbol{p}, \boldsymbol{\lambda}) \right| \leq C r^{-d_G(e, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}.$$

Lemma 27. *The conditional marginal probability of edge e under condition σ_A for $A \subseteq E \backslash e$ in the random cluster model can be translated to the edge-type ratio in the Ising model as*

$$P_{G,e}^{\sigma_A} = \frac{Z_{G-e}^{\text{Ising}, m(\sigma_A)}}{Z_G^{\text{Ising}, m(\sigma_A)}}$$

where $m(\sigma_A) = \{\beta_e \to 1 \mid \sigma_A(e) = -\} \cup \{\beta_e \to \infty \mid \sigma_A(e) = +\}$.

Proof. Pinning an edge e picked or unpicked can also be understood via the modifying on the parameters, as stated in the

$$Z_{G,e}^{\text{RC},+} = p_e Z_G^{\text{RC}}(p_e = 1) \quad \text{and} \quad Z_{G,e}^{\text{RC},-} = (1 - p_e) Z_G^{\text{RC}}(p_e = 0).$$

The corresponding modifying is setting $\beta_e = \infty$ and $\beta_e = 1$ respectively. One can use the rule recursively, let $\boldsymbol{p}^{\sigma_A}$ denote $\boldsymbol{p}$ modified by setting $p_e = 1$ for $\sigma_A(e) = +$ and $p_e = 0$ for $\sigma_A(e) = -$ respectively.

$$P_{G,e}^{\sigma_A} = (1 - p_e) \frac{Z_G(\boldsymbol{p}^{\sigma_A}, p_e = 0)}{Z_G(\boldsymbol{p}^{\sigma_A})}.$$

Then by Lemma 24 and Remark 25,

$$P_{G,e}^{\sigma_A} = \frac{Z_{G-e}^{\text{Ising}, m(\sigma_A)}}{Z_G^{\text{Ising}, m(\sigma_A)}}.$$

□

Thus, the GE-SSM or edge-deletion SSM of the Ising model will directly imply the SSM of the random cluster model.

Theorem 28 (SSM for the random cluster model)**.** *Fix a constant $\delta \in (0, 1)$. For any graph $G = (V, E)$ with parameters $\boldsymbol{p} \in [0, 1]^E$ and $\boldsymbol{\lambda} \in [0, 1 - \delta]^V$, SSM holds for the random cluster model.*

Proof. Following the transformation between the Ising model and the random cluster model, the result follows immediately from Theorem 13. □

4.5 Optimal mixing time on lattice

The mixing time result is a direct consequence of the SSM result in Theorem 28 and the framework in [GS24].

4.5.1 Markov chain and mixing time

Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain over a finite state space Ω with transition matrix P. $(X_t)_{t\in\mathbb{N}}$ is irreducible if for any $x, y \in \Omega$, there exists $t > 0$ such that $P^t(x,y) > 0$. $(X_t)_{t\in\mathbb{N}}$ is aperiodic if for any $x \in \Omega$, $\gcd\{t \in \mathbb{N}^+ \mid P^t(x,x) > 0\} = 1$. A distribution μ over Ω is a stationary distribution of $(X_t)_{t\in\mathbb{N}}$ if $\mu P = \mu$. If the Markov chain is irreducible and aperiodic, then it has a unique stationary distribution. The total variation distance between two distributions μ, ν on the same state space Ω is defined as $d_{\mathrm{TV}}(\mu,\nu) = \max_{S\subseteq\Omega} |\mu(S) - \nu(S)| = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)|$. Suppose μ is the stationary distribution of $(X_t)_{t\in\mathbb{N}}$. The mixing time of the chain is defined as

$$T_{\mathrm{mix}}(\epsilon) = \max_{x_0\in\Omega} \min\{t \in \mathbb{N} \mid d_{\mathrm{TV}}(P^t(x_0,\cdot),\mu) < \epsilon\}.$$

By convention, the standard mixing time is defined as $T_{\mathrm{mix}} = T_{\mathrm{mix}}\left(\frac{1}{4}\right)$.

4.5.2 Glauber dynamics for the random cluster model

The Glauber dynamics for the random cluster model (FK dynamics) is defined as follows. If the configuration at time t is σ, then the configuration at time $t+1$ is obtained by

1. Pick an edge $e \in E$ uniformly at random.
2. Include e in the new configuration with probability $p(\sigma, e) = \frac{\mu(\sigma\cup\{e\})}{\mu(\sigma\cup\{e\})+\mu(\sigma\setminus\{e\})}$, otherwise exclude it.

This dynamics is irreducible and reversible with respect to the distribution $\mu(\cdot)$ induced by the random cluster model, it coverages to the distribution $\pi(\cdot)$ no matter what the initial configuration is. Write $p(\sigma, e)$ explicitly as follows. Suppose $e = (u, v)$ is an edge in the graph. If e is not a cut edge in the configuration $\sigma \cup e$, then $p(\sigma, e) = p_e$. Otherwise, suppose u and v belong to distinct connected components C_1 and C_2 in σ, respectively. Let $x_1 = \prod_{v\in C_1} \lambda_v$ and $x_2 = \prod_{v\in C_2} \lambda_v$. Then,

$$p(\sigma, e) = \frac{p_e(1 + x_1x_2)}{p_e(1 + x_1x_2) + (1 - p_e)(1 + x_1)(1 + x_2)}.$$

If the parameters of the corresponding Ising model satisfy $\beta_e \in [\beta_{\min}, \beta_{\max}]$ and $\lambda_v \in [0, 1]$, where $1 < \beta_{\min} \le \beta_{\max}$, then the following bounds hold: $p(\sigma, e) \le p_e \le 1 - \frac{1}{\beta_{\max}}$ and $p(\sigma, e) \ge \frac{p_e}{6} \ge \frac{1}{6}\left(1 - \frac{1}{\beta_{\min}}\right)$. Thus, $p(\sigma, e)$ is uniformly bounded away from 0 and 1 by a constant distance, i.e., the minimum probability that an edge unchanged in an update step can be determined by $\beta_{\min}$ and $\beta_{\max}$.

4.5.3 Monotonicity and the grand coupling

The grand coupling of the Glauber dynamics can be defined as follows. For a graph $G = (V, E)$, starting from a configuration ω and a boundary condition σ_Λ, the grand coupling is given by $\{X^{\omega}_{t,\sigma_\Lambda}\}$, indexed by the initial configuration ω (or a distribution) and the boundary condition σ_Λ. We assign a Poisson clock of rate 1 to each edge $e \in E$. When the clock for an edge e rings at time t, we sample a random variable U_t uniformly from $[0, 1]$. If $e \in \Lambda$, the configuration remains unchanged; otherwise, we update the configuration according to the Glauber dynamics: if $U_t \ge 1 - p(\sigma, e)$, we include e in the configuration; otherwise, we exclude e.

Similarly to the standard random cluster model, the grand coupling of the weighted random cluster model is monotonic.

Lemma 29. *[FGW23, Lemma 8.2] Suppose $0 \le p_e < 1$ for all $e \in E$ and $0 \le \lambda_v \le 1$ for all $v \in V$. Then the grand coupling of the Glauber dynamics for the weighted random cluster model is monotonic.*

The key of lemma is the following inequality, suppose $\sigma_1 \le \sigma_2$, then $p(\sigma_1, e) \le p(\sigma_2, e)$. The monotonic grand coupling also implies the monotonicity of the Glauber dynamics.

4.5.4 Strong spatial mixing implies optimal mixing time

Follow the approach used for the standard random cluster model on lattices in [GS24]. They consider the continuous Glauber dynamics and thus need a constant minimum probability that an edge is unchanged, which can be realized by setting $\beta_{\min}$ and $\beta_{\max}$. Since the weighted random cluster model still exhibits the monotonicity property and the monotonic grand coupling, their proof also applies, showing that strong spatial mixing in the weighted random cluster model implies the optimal mixing time of the Glauber dynamics.

Theorem 30. *Fix $1 < \beta_{\min} \le \beta_{\max}$, $\delta \in (0, 1)$, and $d \in \mathbb{N}$. For any subgraph $G = (V, E)$ of the infinite d-dimensional lattice, with parameters $\boldsymbol{\beta} \in [\beta_{\min}, \beta_{\max}]^E$ and $\boldsymbol{\lambda} \in [0, 1-\delta]^V$, the mixing time of the Glauber dynamics for the corresponding random-cluster representation of the Ising model is $O(m \log m)$, where $m = |E|$.*

5 LDC and SSM for Other Models

5.1 SSM for hypergraph independence polynomial

A hypergraph $H = (V, E)$ is a set of vertices V along with a set of edges, where each edge is a nonempty subset of V. The degree of a vertex v in a hypergraph is the number of edges containing v. An independent set in H is a set of vertices $I \subseteq V$ such that no edge in E is a subset of I. Let $\mathcal{I}$ be the set of all independent sets in H, then the independence polynomial of H is defined as

$$Z_H(\lambda) = \sum_{I \in \mathcal{I}} \lambda^{|I|}.$$

We continue using the notations $+$ and $-$ to denote the vertex being in and out of the independent set, respectively. A partial configuration σ_Λ is feasible if it is an independent set. We say v is proper to σ_Λ if $v \notin \Lambda$ and $\sigma^+_{\Lambda \cup \{v\}}$ is feasible.

We need to define some operations on the hypergraph $H = (V, E)$.

1. **Induced sub-hypergraph.** For $\Lambda \subseteq V$, denote $H_\Lambda = (\Lambda, \{e \cap \Lambda : e \in E, e \cap \Lambda \neq \varnothing\})$.
2. **Mild vertex deletion.** For $v \in V$, denote $H \ominus v = H_{V \setminus \{v\}}$. For $S \subseteq V$, denote $H \ominus S = H_{V \setminus S}$.
3. **Total vertex deletion.** For $v \in V$, denote $H \backslash v = (V \backslash \{v\}, \{e \in E : v \notin e\})$. For $S \subseteq V$, denote $H \backslash S = (V \backslash S, \{e \in E : e \cap S = \varnothing\})$.

Note that $H \ominus v$ retains as many edges as possible while $H \backslash v$ removes all edges containing v. In fact, we have the following relations [Tri16]:

$$Z^+_{H,v}(\lambda) = \lambda Z_{H \ominus v}(\lambda) \quad \text{and} \quad Z^-_{H,v}(\lambda) = Z_{H \backslash v}(\lambda).$$

Note $H \ominus v$ reverse the edges containing v as possible while $H\backslash v$ remove all edges containing v. Then one can see that $Z^{+}_{H,v}(\lambda) = \lambda Z_{H\ominus v}(\lambda)$ and $Z^{-}_{H,v}(\lambda) = Z_{H\backslash v}(\lambda)$. The marginal probability that v is in an independent set is defined as

$$P_{H,v}(\lambda) = \frac{Z^{+}_{H,v}(\lambda)}{Z_H(\lambda)} = \frac{\lambda Z_{H\ominus v}(\lambda)}{Z_H(\lambda)}.$$

Similarly, the conditional probability that v is in an independent set, given a partial configuration σ_Λ, is defined as

$$P^{\sigma_\Lambda}_{H,v}(\lambda) = \frac{Z^{\sigma_\Lambda,+}_{H,v}(\lambda)}{Z^{\sigma_\Lambda}_{H}(\lambda)}.$$

For a partial configuration σ_Λ that pins vertices in Λ^+ to $+$ and vertices in Λ^- to $-$, where (Λ^+, Λ^-) is a partition of Λ, denote $H[\sigma_\Lambda] = (H \ominus \Lambda^+)\backslash\Lambda^-$. Then we have the following identity:

$$P^{\sigma_\Lambda}_{H,v}(\lambda) = P_{H[\sigma_\Lambda],v}(\lambda),$$

which allows us to analyze the ratio without the partial configuration. This identity can be verified by comparing each independent set in H with σ_Λ to those in $H[\sigma_\Lambda]$.

Denote $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$ and $\lambda_s(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}}$. We directly state the strong spatial mixing result for the hypergraph independence polynomial as a theorem, which is stronger than the definition in [BGG+19] and matches the result for graphs in [Reg23].

Theorem 31 (SSM)**.** *Fix $\Delta \geq 3$ and $\lambda \in \mathbb{D}_{\lambda_s(\Delta)} \cup (0, \lambda_c(\Delta))$. There exist constants $C > 0$ and $r > 1$ such that for any hypergraph $H = (V, E)$ with maximum degree at most Δ, any two feasible partial configurations σ_{Λ_1} and τ_{Λ_2} where Λ_1 may be different with Λ_2, and any vertex v proper to σ_{Λ_1} and τ_{Λ_2}, we have*

$$|P^{\sigma_{\Lambda_1}}_{H,v}(\lambda) - P^{\tau_{\Lambda_2}}_{H,v}(\lambda)| \leq Cr^{-d_H(v,\Lambda_1 \neq \Lambda_2)}$$

where $\Lambda_1 \neq \Lambda_2$ is the set $(\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1) \cup \{v \in \Lambda_1 \cap \Lambda_2 : \sigma_{\Lambda_1}(v) \neq \tau_{\Lambda_2}(v)\}$.

5.1.1 LDC

In [Reg23], Regts establishes the LDC for the hardcore model (the independence polynomial of a graph) via cluster expansion techniques. However, the result is not immediately clear for general hypergraphs. Our technique for establishing divisibility also extends to hypergraphs. By constructing a bijection, we prove the following divisibility lemma, which subsequently leads to the LDC.

Lemma 32. *Let $H = (V, E)$ be a hypergraph, σ_Λ be a partial configuration on $\Lambda \subseteq V$, u, v be two distinct vertices in $V\backslash\Lambda$, then*

$$\lambda^{d_H(u,v)+1} \mid Z^{\sigma_\Lambda,+,+}_{H,u,v}(\lambda) Z^{\sigma_\Lambda,-,-}_{H,u,v}(\lambda) - Z^{\sigma_\Lambda,+,-}_{H,u,v}(\lambda) Z^{\sigma_\Lambda,-,+}_{H,u,v}(\lambda)$$

Proof. Let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ denotes the sets of independent sets admitting $Z^{\sigma_\Lambda,+,+}_{H,u,v}(\lambda)$, $Z^{\sigma_\Lambda,-,-}_{H,u,v}(\lambda)$, $Z^{\sigma_\Lambda,+,-}_{H,u,v}(\lambda)$ and $Z^{\sigma_\Lambda,-,+}_{H,u,v}(\lambda)$ respectively. Then

$$\begin{aligned}&Z^{\sigma_\Lambda,+,+}_{H,u,v}(\lambda) Z^{\sigma_\Lambda,-,-}_{H,u,v}(\lambda) - Z^{\sigma_\Lambda,+,-}_{H,u,v}(\lambda) Z^{\sigma_\Lambda,-,+}_{H,u,v}(\lambda)\\ =&\sum_{I_1 \in \mathcal{I}_1} \sum_{I_2 \in \mathcal{I}_2} \lambda^{|I_1|+|I_2|} - \sum_{I \in \mathcal{I}_3} \sum_{I \in \mathcal{I}_4} \lambda^{|I_3|+|I_4|}\end{aligned}$$

Let A be the set of (I_1, I_2) such that $|I_1|+|I_2| < d_H(u,v)+1$ and B be the set of (I_3, I_4) such that $|I_3|+|I_4| < d_H(u,v)+1$. We will construct a bijection between A and B and if $(I_3, I_4) = f(I_1, I_2)$, then $|I_1| + |I_2| = |I_3| + |I_4|$. For any $(I_1, I_2) \in A$, since $|I_1| + |I_2| < d_H(u,v) + 1$, u and v are disconnected in the induced sub-hypergraph $H_{I_1 \cup I_2}$. Let S be the connected component in $H_{I_1 \cup I_2}$ containing u and $T = V \backslash S$. Then $(I_3, I_4) = (I_1|_S \cup I_2|_T, I_1|_T \cup I_2|_S) \in B$. Certainly $|I_3| + |I_4| = |I_1| + |I_2|$ and the operation is reversible since $I_3 \cup I_4 = I_1 \cup I_2$ is unchanged after the swap. $\square$

The divisibility relation directly implies the so-called point-to-point LDC in [SY24]. Moreover, by induction on $|\sigma_{\Lambda_1} \neq \sigma_{\Lambda_2}|$, the point-to-point LDC implies the LDC.

Lemma 33 (Point-to-point LDC)**.** *Let $H = (V, E)$ be a hypergraph, $\sigma_{\Lambda_1}, \sigma_{\Lambda_2}$ be two partial configurations on $\Lambda_1, \Lambda_2 \subseteq V$, v be a proper vertex to σ_{Λ_1} and σ_{Λ_2}, then*

$$\lambda^{d_H(v,u)+1} \mid P_{H,v}^{\sigma_{\Lambda_1}}(\lambda) - P_{H,v}^{\sigma_{\Lambda_1},u^+}(\lambda) \quad \textit{and} \quad \lambda^{d_H(v,u)+1} \mid P_{H,v}^{\sigma_{\Lambda_1}}(\lambda) - P_{H,v}^{\sigma_{\Lambda_1},u^-}(\lambda).$$

Lemma 34 (LDC)**.** *Let $H = (V, E)$ be a hypergraph, $\sigma_{\Lambda_1}, \sigma_{\Lambda_2}$ be two partial configurations on $\Lambda_1, \Lambda_2 \subseteq V$, v be a proper vertex to σ_{Λ_1} and σ_{Λ_2}, then*

$$\lambda^{d_H(v,\Lambda_1 \neq \Lambda_2)+1} \mid P_{H,v}^{\sigma_{\Lambda_1}}(\lambda) - P_{H,v}^{\sigma_{\Lambda_2}}(\lambda).$$

5.1.2 Uniform bound

Galvin and coauthors claim that the independence polynomial of a hypergraph with maximum degree Δ is zero-free in the disk $\mathbb{D}_{\lambda_s(\Delta+1)}$ [GMP$^+$24]. Later, Bencs and Buys [BB23] improve the zero-free region to $\mathbb{D}_{\lambda_s(\Delta)}$ and provide another zero-free region around the Shearer's bound $(0, \lambda_c(\Delta))$, extending the result from graphs to hypergraphs as shown in [PR19]. With the zero-free region, we can extend the result in [Reg23] of the graph independence polynomial to hypergraphs.

Lemma 35 (Theorem 1.1 in [BB23])**.** *Let $\Delta \geq 2$. For any hypergraph $H = (V, E)$ with maximum degree at most Δ and $\boldsymbol{\lambda} \in \mathbb{C}^V$ with $|\lambda_v| \leq \lambda_s(\Delta)$ for all $v \in V$ we have $Z_H(\lambda) \neq 0$.*

Lemma 36 (Theorem 1.2 in [BB23])**.** *Let $\Delta \geq 3$. There exists an open neighborhood U_Δ of the interval $(0, \lambda_c(\Delta))$ such that for any hypergraph $H = (V, E)$ with maximum degree at most Δ and $\lambda \in U$ we have $Z_H(\lambda) \neq 0$.*

Lemma 37. *Let $\Delta \geq 3$, H be a hypergraph with maximum degree at most Δ, σ_Λ be a partial configuration on $\Lambda \subseteq V$, v be a proper vertex to σ_Λ. If $\lambda \in (\mathbb{D}_{\lambda_s(\Delta)} \cup U_\Delta) \backslash \{0\}$, then $P_{H,v}^{\sigma_\Lambda}(\lambda)$ avoids 0 and 1.*

Proof. Since $P_{H,v}^{\sigma_\Lambda}(\lambda) = P_{H[\sigma_\Lambda],v}(\lambda)$, prove $P_{H,v}(\lambda)$ always avoids 0 and 1 is enough. By Lemmas 35 and 36, $Z_H(\lambda), Z_{H\backslash v}(\lambda)$ and $Z_{H\ominus v}(\lambda) \neq 0$. Thus $P_{H,v}(\lambda) = \frac{\lambda Z_{H\ominus v}(\lambda)}{Z_H(\lambda)} \neq 0$ and $P_{H,v}(\lambda) = 1 - \frac{Z_{H\backslash v}(\lambda)}{Z_H(\lambda)} \neq 1$. We are done. $\square$

Lemma 38 (uniform bound)**.** *Fix $\Delta \geq 3$, let $U = (\mathbb{D}_{\lambda_s(\Delta)} \cup U_\Delta) \backslash \{0\}$, S be a compact subset of U. There exist a constant $C > 0$ such that for any hypergraph $H = (V, E)$ with maximum degree at most Δ, any $v \in V$, any $\lambda \in S$, we have $|P_{H,v}(\lambda)| \leq C$.*

Proof. By Lemma 37, $P_{H,v}(\lambda)$ always avoids 0 and 1 for $\lambda \in U$. Pick $\lambda' \in (0, \lambda_s(\Delta))$, then $P_{H,v}(\lambda')$ is a probability and hence contained in $[0, 1]$. Then by Lemma 6, the upper bound is got. $\square$

If the zero-free region is not a disk, it seems that we cannot apply Lemma 16 to deduce that any fixed λ in the zero-free region exhibits SSM. However, by the Riemann mapping theorem, we can transform an arbitrary zero-free region into a unit disk and then apply Lemma 16, where the LDC and uniform bound still hold. For details, see [Reg23, SY24].

Proof of Theorem 31. This follows from the argument of the Riemann mapping theorem and the results of Lemmas 16, 34 and 38. □

5.1.3 FPTAS

The computation tree of the hypergraph independence polynomial introduced in [LL14, LYZ15] is the key tool to derive FPTAS from the SSM property. We don't give the exact construction of the computation tree here, but utilize it as a black box.

Theorem 39. *Let $H = (V, E)$ be a hypergraph of maximum degree Δ. Then there exists a hypertree T with root v and maximum degree at most Δ such that $P_{H,v}(\lambda) = P_{T,v}(\lambda)$. If size of edges in H is at most k, let T_k be the truncation of T at depth d from v, then we can compute $P_{T_d,v}(\lambda)$ exactly in time $O(|V|(k\Delta)^d)$.*

Lemma 40. *If H is a hypergraph, v is a vertex in H and $\lambda > 0$, then $0 \leq P_{H,v}(\lambda) \leq \frac{\lambda}{1+\lambda}$.*

Proof. If I is an independent set in H containing v with weight w, then $I \backslash \{v\}$ is still an independent set in H with weight w/λ. Thus $Z_H(\lambda) = Z^+_{H,v}(\lambda) + Z^-_{H,v}(\lambda) \geq Z^+_{H,v}(\lambda)(1 + 1/\lambda)$, which implies $P_{H,v}(\lambda) \leq \frac{\lambda}{1+\lambda}$. □

Lemma 41. *Fix $\Delta \geq 3$ and S be a compact subset of $(\mathbb{D}_{\lambda_s(\Delta)} \cup U_\Delta)\backslash\{0\}$, there exists a constant $C > 0$ such that for any hypergraph $H = (V, E)$ with maximum degree at most Δ, any $v \in V$, any $\lambda \in S$, we have $|1 - P_{H,v}(\lambda)| \geq C$.*

Proof. Note $P_{H,v}(\lambda)$ always avoids 0 and 1 for $\lambda \in (\mathbb{D}_{\lambda_s(\Delta)} \cup U_\Delta)\backslash\{0\}$. Then $\frac{1}{1-P_{H,v}(\lambda)}$ is analytic for $\lambda \in (\mathbb{D}_{\lambda_s(\Delta)} \cup U_\Delta)\backslash\{0\}$ and always avoids 0 and 1. And pick a positive constant $\lambda' \in (0, \lambda_s(\Delta))$, then $0 \leq P_{H,v}(\lambda') \leq \frac{\lambda'}{1+\lambda'}$ always holds, i.e. $1 \leq \frac{1}{1-P_{H,v}(\lambda)} \leq 1 + \lambda'$. Then by Lemma 6, the upper bound of $\left|\frac{1}{1-P_{H,v}(\lambda)}\right|$ is got, and then the lower bound of $|1 - P_{H,v}(\lambda)|$ is got. □

Theorem 42 (FPTAS)**.** *Fix $\Delta \geq 3$, $k \geq 2$ and $\lambda \in \mathbb{D}_{\lambda_s(\Delta)} \cup U_\Delta$, there exists an FPTAS for the hypergraph independence polynomial $Z_H(\lambda)$ for any hypergraph $H = (V, E)$ with maximum degree at most Δ and maximum edge size at most k.*

Proof. When $\lambda = 0$, the problem is trivial. Consider $\lambda \neq 0$. Write $V = \{v_1, \ldots, v_n\}$, let $\Lambda_i = \{v_1, \ldots, v_i\}$ and σ_i be the partial configuration which maps all vertices in Λ_i to $-$ (for $i = 0, \ldots, n$). Then

$$\frac{1}{Z_H(\lambda)} = \frac{Z_H^{\sigma_n}(\lambda)}{Z_H^{\sigma_0}(\lambda)} = \prod_{i=1}^{n} \frac{Z_H^{\sigma_i}(\lambda)}{Z_H^{\sigma_{i-1}}(\lambda)} = \prod_{i=1}^{n} \left[1 - P_{H,v_i}(\lambda)\right] = \prod_{i=1}^{n} \left[1 - P_{H[\sigma_{i-1}],v_i}(\lambda)\right].$$

To approximate $Z_H(\lambda)$ with factor ε, approximating $1 - P_{H[\sigma_{i-1}],v_i}(\lambda)$ with factor $\frac{\varepsilon}{n}$ is enough. By Lemma 41, additive error $\frac{C\varepsilon}{2n}$ for some constant $C > 0$ is enough. Then by computation tree in [LL14] and the SSM result, truncating the computation tree at depth $O(\log \frac{n}{\varepsilon})$ (one way is pinning all vertices at $O(\log \frac{n}{\varepsilon})$ depth to $(-)$) and using the SSM result, the running time is $\mathrm{poly}(\frac{n}{\varepsilon})$, thus we can get the FPTAS. □

Remark 43. *In [LL14, LYZ15], the authors prove a computationally efficient correlation decay for $\lambda \in (0, \lambda_c(\Delta))$, which leads to a faster decay rate when dealing with hyperedges of larger sizes. Then they derive an FPTAS for hypergraphs with bounded degree but unbounded edge size based on this correlation decay result.*

5.2 SSM for binary symmetric Holant problems

Let $G = (V, E)$ be a graph of maximum degree Δ. We consider the Holant problem in the binary symmetric case, which we now describe. Let $\{f_v\}_{v \in V} : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a family of functions, one for each vertex $v \in V$ in the input graph. One should think of each f_v as representing a local constraint on the assignments to edges incident to v. Since we are restricting ourselves to the binary case, our configurations σ will map edges to $\{0, 1\}$ (or $-$ and $+$ spins). Furthermore, since we are restricting ourselves to the symmetric case, our local functions f_v will only depend on the number of edges incident to v which are mapped to 1. With these $\{f_v\}_{v \in V}$ in hand, we may write the multivariate partition function as

$$Z_G(\boldsymbol{\lambda}) = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(|\sigma_{E(v)}|) \prod_{e \in E, \sigma(e) = 1} \lambda_e,$$

where $E(v)$ is the set of all edges adjacent to v, $\sigma_{E(v)}$ is the configuration restricted on $E(v)$, and $|\sigma_{E(v)}|$ is the number of edges in $E(v)$ with assignment 1.

This class of problems is already incredibly rich and encompasses many classical objects studied in combinatorics and statistical physics. As stated in [CLV24], the weighted even subgraphs model and the Ising model on line graphs are included for certain choices of f_v.

- *Weighted Even Subgraphs:* In this case, all f_v are the same and given by the weighted "parity" function. More specifically, for a fixed positive parameter $\rho > 0$, we have

$$f_v(k) = \begin{cases} 1, & \text{if } k \text{ is even;} \\ \rho, & \text{if } k \text{ is odd.} \end{cases}$$

  In the case $\rho = 0$, then $Z_G(\mathbf{1})$ counts the number of even subgraphs, that is, subsets of edges such that all vertices have even degrees in the resulting subgraph.

- *Ising Model on Line Graphs:* In this case, each f_v depends on the degree of v. If $\beta > 0$ is some fixed parameter (independent of v), and $d = \deg(v)$, then we have

$$f_v(k) = \begin{cases} \beta^{\binom{k}{2}} \beta^{\binom{d-k}{2}}, & \text{if } 0 \leq k \leq d; \\ 0, & \text{otherwise.} \end{cases}$$

Let $G = (V, E)$ be a graph, $e \in E$ an edge, and σ_Λ a partial configuration on $\Lambda \subseteq E \backslash \{e\}$. Similarly to the random cluster model, the conditional probability that e is pinned $+$ is given by $P_{G,e}^{\sigma_\Lambda}(\lambda) = Z_{G,e}^{\sigma_\Lambda,+}(\lambda) / Z_G^{\sigma_\Lambda}(\lambda)$. The strong spatial mixing can also be defined as below.

Definition 44 (SSM for Holant)**.** *Fix f_v be the local function and λ. Let $\mathcal{G}$ be a family of graphs. The Holant problem defined on $\mathcal{G}$ with f_v and λ is said to satisfy strong spatial mixing with exponential rate $r > 1$ if there exists a constant C such that for any $G = (V, E) \in \mathcal{G}$, any edge $e \in E$, any partial configuration σ_{Λ_1} and τ_{Λ_2} where $\Lambda_1, \Lambda_2 \subseteq E \backslash e$, we have*

$$\left| P_{G,e}^{\sigma_{\Lambda_1}}(\lambda) - P_{G,e}^{\tau_{\Lambda_2}}(\lambda) \right| \leq C r^{-d_G(e, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}.$$

5.2.1 Zerofree

In [CLV24], to establish the spectral independence property from zero-freeness, the authors prove the zero-free region for weighted even subgraphs and the Ising model on line graphs. Notably, in [CLV24], the authors exclude the λ factor when pinning a vertex to + (or 1), whereas we do not. Consequently, we exclude the point 0 from their zero-free region.

Lemma 45. *Fix $\rho \in (0,1)$ and $\Delta \in \mathbb{N}^+$, then there exists a complex region U containing $[0,\infty)$ such that for all graphs G with bounded degree Δ and all partial configurations σ, the partition function of the weighted even subgraph model satisfies $Z_G(\lambda) \neq 0$ for any $\lambda \in U \setminus \{0\}$.*

Lemma 46. *Fix $\beta \in (0,1)$ and $\Delta \in \mathbb{N}^+$, then there exists a complex region U containing $[0,\infty)$ such that for all line graphs G with bounded degree Δ and all partial configurations σ, the partition function of the antiferromagnetic Ising model satisfies $Z_G(\lambda) \neq 0$ for any $\lambda \in U \setminus \{0\}$.*

Note that $P^{\sigma_\Lambda}_{G,e}$ is analytic in the region $\lambda \in U$. One only needs to check that the ratio is well defined at $\lambda = 0$. This holds because $Z^{\sigma_\Lambda,+}_{G,e}$ has a higher order of λ than $Z^{\sigma_\Lambda}_{G}$.

5.2.2 LDC

Lemma 47. *Let $G = (V,E)$ be a graph, σ be a partial configuration on $\Lambda \subseteq E$, e_1 and e_2 be two different edges in $E \backslash \Lambda$, then*

$$\lambda^{d_G(e_1,e_2)+2} \mid Z^{\sigma,+,+}_{G,e_1,e_2} Z^{\sigma,-,-}_{G,e_1,e_2} - Z^{\sigma,+,-}_{G,e_1,e_2} Z^{\sigma,-,+}_{G,e_1,e_2}.$$

Proof. Let S_1 be the set of configurations that agree with $Z^{\sigma,+,+}_{G,e_1,e_2}$, and similarly define S_2, S_3 and S_4. Then, we have

$$\begin{aligned}
&Z^{\sigma,+,+}_{G,e_1,e_2} Z^{\sigma,-,-}_{G,e_1,e_2} - Z^{\sigma,+,-}_{G,e_1,e_2} Z^{\sigma,-,+}_{G,e_1,e_2} \\
=& \sum_{\sigma_1 \in S_1, \sigma_2 \in S_2} w(\sigma_1) w(\sigma_2) - \sum_{\sigma_3 \in S_3, \sigma_4 \in S_4} w(\sigma_3) w(\sigma_4)
\end{aligned}$$

Define $A = \{(\sigma_1,\sigma_2) \mid n_+(\sigma_1) + n_+(\sigma_2) \leq d_G(e_1,e_2) + 1, \sigma_1 \in S_1, \sigma_2 \in S_2\}$ and similarly define $B \subseteq S_3 \times S_4$. We show there exists a bijection $f : A \to B$ such that if $(\sigma_3,\sigma_4) = f(\sigma_1,\sigma_2)$ then $w(\sigma_1)w(\sigma_2) = w(\sigma_3)w(\sigma_4)$. If $n_+(\sigma_1) + n_+(\sigma_2) \leq d_G(e_1,e_2) + 1$, then the subgraph $H = (V, \sigma_1|\sigma_2)$ is disconnected. Pick S as the connected component containing e_1, and let $T = E \backslash S$. Define $(\sigma_3,\sigma_4) = (\sigma_1|_S \cup \sigma_2|_T, \sigma_2|_S \cup \sigma_1|_T)$, then $f(\sigma_1,\sigma_2) = (\sigma_3,\sigma_4)$ satisfies our requirements. Firstly f is bijection since $\sigma_1|\sigma_2 = \sigma_3|\sigma_4$, the process is fully reversible. Since there are no + edge between S and T in $\sigma_i (i = 1,2,3,4)$, the local functions f_v for each $v \in S$ are determined by $\sigma_i[S]$ and similarly for $v \in T$. Thus,

$$\begin{aligned}
w_G(\sigma_1) w_G(\sigma_2) =& w_{G[S]}(\sigma_1|_S) w_{G[T]}(\sigma_1|_T) w_{G[S]}(\sigma_2|_S) w_{G[T]}(\sigma_2|_T) \\
=& w_{G[S]}(\sigma_1|_S) w_{G[T]}(\sigma_2|_T) w_{G[S]}(\sigma_2|_S) w_{G[T]}(\sigma_1|_T) \\
=& w_{G[S]}(\sigma_3|_S) w_{G[T]}(\sigma_3|_T) w_{G[S]}(\sigma_4|_S) w_{G[T]}(\sigma_4|_T) \\
=& w_G(\sigma_3) w_G(\sigma_4).
\end{aligned}$$

□

The divisibility relation implies point-to-point LDC, which then extends to LDC by induction. The definition of LDC in the Holant framework follows the same description as in the random cluster model.

5.2.3 SSM

Similar to before, pick any $\lambda > 0$ is a uniformly bounded point (as a probability), then we can deduce the uniform bound on a compact subset by Lemma 6 from the zero-freeness result. Then following Regts's approach, we can establish the SSM property for binary symmetric Holant problems once the zero-free region is well understood.

Theorem 48. *Fix $\rho \in (0,1)$, $\Delta \in \mathbb{N}^+$ and $\lambda > 0$. Then weighted even subgraph model for all graphs G with bounded degree Δ exhibits SSM.*

Theorem 49. *Fix $\beta \in (0,1)$, $\Delta \in \mathbb{N}^+$ and $\lambda > 0$. Then the Ising model for all line graphs G with bounded degree Δ exhibits SSM.*

5.3 Edge-type SSM for Potts model

The partition function of the Potts model (without external field) of a graph $G = (V, E)$ is defined as

$$Z_G(q, \boldsymbol{w}) = \sum_{\sigma: V \to [q]} \prod_{\substack{(u,v) \in E \\ \sigma(u) = \sigma(v)}} w_{(u,v)}.$$

where $[q] = \{1, 2, \ldots, q\}$, q is the number of colors, $\boldsymbol{w} = (w_e)_{e \in E}$ is the edge activity vector. In the univariate case, write $w = 1 + z$, then the partition function of the Potts Model can be written in the form of the Tutte polynomial [S+05] as

$$Z_G(q, w) = \sum_{F \subseteq E} q^{\kappa(V,F)} z^{|F|},$$

where $\kappa(V, F)$ is the number of connected components of the spanning subgraph (V, F).

Similar to the edge-type SSM for the Ising model in Corollary 15, define the ratio of the partition function of the Potts model as

$$P_{G,e}(w) = \frac{Z_{G-e}(q, w)}{Z_G(q, w)}.$$

We can prove the Potts model exhibits edge-deletion SSM, where the constant $\eta \geq 0.002$ is from the zero-free region [BBR24].

Theorem 50. *Fix $\Delta \in \mathbb{N}$, $q \geq (2 - \eta)(2\Delta - 2)$ and $w \in [0, 1]$, then there exist constant $C > 0$ and $r > 1$ such that for any graph $G = (V, E)$ with maximum degree at most Δ, $e \in E$, $A, B \subseteq E \backslash \{e\}$, we have*

$$|P_{G-A,e}(q, w) - P_{G-B,e}(q, w)| \leq C r^{-d_G(e, A \neq B)}.$$

In [BBR24], the zero-free region of the univariate Potts model is studied, and the authors claimed that it also works in the multivariate setting.

Lemma 51 (Theorem 1 and Section 8 in [BBR24])**.** *There exists a constant $\eta \geq 0.002$ such that for all integers $\Delta \geq 3$ and $q \geq (2 - \eta)\Delta$ there exists an open set $U_{\Delta,q} \subseteq \mathbb{C}$ containing the interval $[0, 1]$ such that for any graph $G = (V, E)$ of maximum degree at most Δ and $\boldsymbol{w} \in (U_{\Delta,q})^E$ and we have $Z_G(q, \boldsymbol{w}) \neq 0$.*

By Lemma 51, we can get the following result. For $A, B \subseteq \mathbb{C}$, define $A \cdot B = \{ab \mid a \in A, b \in B\}$. We can immediately get there exist an open set $\mathcal{U}_{\Delta,q} \subseteq U_{\Delta,q}$ containing the real *closed* interval $[0, 1]$ and $\mathcal{U}_{\Delta,q} \cdot \mathcal{U}_{\Delta,q} \subseteq U_{\Delta,q}$. Open set $\mathcal{U}_{\Delta,q} \backslash \{1\}$ will guarantee the ratio $P_{G,e}(w) = \frac{Z_{G-e}(q,w)}{Z_G(q,w)}$ avoid 0 and 1.

5.3.1 Uniform bound

Lemma 52. *If $\Delta \in \mathbb{N}$, $q \geq (2-\eta)(2\Delta-2)$, $w \in \mathcal{U}_{2\Delta-2,q}\backslash\{1\}$, $G=(V,E)$ is a graph with maximum degree at most Δ and $e \in E$, then $P_{G,e}(w)$ avoid 0 and 1.*

Proof. By Lemma 51, $P_{G,e}(w) \neq 0$ is trivial. We prove $P_{G,e}(w) \neq 1$. Let $e=(u,v)$, then

$$\begin{aligned}
&Z_G(q,w) - Z_{G-e}(q,w) \\
=&\sum_{\sigma\in[q]^V} \prod_{\substack{(x,y)\in E\\ \sigma(x)=\sigma(y)}} w - \sum_{\sigma\in[q]^V} \prod_{\substack{(x,y)\in E-e\\ \sigma(x)=\sigma(y)}} w \\
=&(w-1) \sum_{\substack{\sigma\in[q]^V\\ \sigma(u)=\sigma(v)}} \prod_{\substack{(x,y)\in E-e\\ \sigma(x)=\sigma(y)}} w.
\end{aligned}$$

Thus we can construct $G'=(V',E')$ from G by merging u,v into a single vertex x, if parallel edges (u,y) and (v,y) exist in G, merge them into a single edge and set $w_{(x,y)} = w_{(u,y)}w_{(v,y)}$. Then $Z_G(q,w) - Z_{G-e}(q,w) = (w-1)Z_{G'}(q,\boldsymbol{w})$, where $\boldsymbol{w}$ is the edge activity vector of G'. Note $\boldsymbol{w} \in (U_{2\Delta-2,q})^{E'}$ and G' has maximum degree at most $2\Delta-2$, since $q \geq (2-\eta)(2\Delta-2)$, by Lemma 51, $Z_{G'}(q,\boldsymbol{w}) \neq 0$ and hence $P_{G,e}(w) \neq 1$. □

Pick a small enough $\varepsilon > 0$ such that $1+\varepsilon \in \mathcal{U}_{2\Delta-2,q}$, one can see that $0 < P_{G,e}(1+\varepsilon) < 1$ always holds. Then by Lemma 6, we can get the uniform bound of the ratio of the partition function of the Potts model.

Lemma 53. *Fix $\Delta \in \mathbb{N}$, and S is a compact subset of $\mathcal{U}_{2\Delta-2,q}\backslash\{1\}$. There exists a constant $C > 0$ such that for any graph $G=(V,E)$ with maximum degree at most Δ, any $q \geq (2-\eta)(2\Delta-2)$, any $e \in E$, any $w \in S$, we have $|P_{G,e}(w)| \leq C$.*

5.3.2 LDC

Lemma 54. *Let $G=(V,E)$ be a graph, e_1,e_2 be two different edges in G, then*

$$(w-1)^{d_G(e_1,e_2)} \mid Z_{G-e_1}(q,w)Z_{G-e_2}(q,w) - Z_G(q,w)Z_{G-\{e_1,e_2\}}(q,w).$$

Proof. Let $z = w-1$, then

$$\begin{aligned}
&Z_{G-e_1}(q,w)Z_{G-e_2}(q,w) - Z_G(q,w)Z_{G-\{e_1,e_2\}}(q,w) \\
=&\sum_{\substack{F_1\subseteq E-e_1,\\ F_2\subseteq E-e_2}} q^{\kappa(V,F_1)+\kappa(V,F_2)} z^{|F_1|+|F_2|} - \sum_{\substack{F_3\subseteq E,\\ F_4\subseteq E-\{e_1,e_2\}}} q^{\kappa(V,F_3)+\kappa(V,F_4)} z^{|F_3|+|F_4|}
\end{aligned}$$

Let A be the set of (F_1,F_2) in the first sum such that $|F_1|+|F_2| < d_G(e_1,e_2)$ and B be the set of (F_3,F_4) in the second sum such that $|F_3|+|F_4| < d_G(e_1,e_2)$. We will show that there exists a bijection f between A and B such that if $(F_3,F_4) = f(F_1,F_2)$, then $|F_3|+|F_4| = |F_1|+|F_2|$ and $\kappa(V,F_3)+\kappa(V,F_4) = \kappa(V,F_1)+\kappa(V,F_2)$.

Let F_1,F_2 be a pair in A, since $|F_1|+|F_2| < d_G(e_1,e_2)$, then e_1,e_2 are disconnected in the subgraph $H=(V,F_1\cup F_2\cup\{e_1,e_2\})$. Consider the connected component S of H, which contains e_1, and let $T = V\backslash S$. Then $F_3 = F_1|_T \cup F_2|_S$ and $F_4 = F_1|_S \cup F_2|_T$ are in B. One can check that (F_3,F_4) is the desired pair and the process is reversible (since $F_1\cup F_2 = F_3\cup F_4$). We are done. □

Lemma 55. *Let $G = (V, E)$ be a graph, $e \in E$, and $A \subseteq E \backslash \{e\}$, then the Taylor series near $w = 1$ of $P_{G,e}(q, w)$ and $P_{G-A,e}(q, w)$ satisfies*

$$(w - 1)^{d_G(e,A)} \mid P_{G,e}(q, w) - P_{G-A,e}(q, w).$$

Proof. We prove this by induction on $|A|$. The base case $|A| = 1$, for instance $A = \{e'\}$,

$$\begin{aligned} P_{G,e}(q, w) - P_{G-e',e}(q, w) =& \frac{Z_{G-e}(q, w)}{Z_G(q, w)} - \frac{Z_{G-\{e,e'\}}(q, w)}{Z_{G-e'}(q, w)} \\ =& \frac{Z_{G-e}(q, w)Z_{G-e'}(q, w) - Z_G(q, w)Z_{G-\{e,e'\}}(q, w)}{Z_G(q, w)Z_{G-e'}(q, w)}. \end{aligned}$$

Clearly $\frac{1}{Z_G(q,w)Z_{G-e'}(q,w)}$ is analytic near $w = 1$. By Lemma 54, we have $(w - 1)^{d_G(e,e')} \mid P_{G,e}(q, w) - P_{G-e',e}(q, w)$.

Now consider the case $k \geq 2$, suppose the statement holds for $|A| \leq k - 1$, we prove it for $|A| = k$. Pick $e' \in A$, let $A' = A \backslash \{e'\}$, then

$$P_{G,e}(q, w) - P_{G-A,e}(q, w) = [P_{G,e}(q, w) - P_{G-A',e}(q, w)] + [P_{G-A',e}(q, w) - P_{G-A,e}(q, w)].$$

By induction hypothesis, we have $(w-1)^{d_G(e,A')} \mid P_{G,e}(q, w)-P_{G-A',e}(q, w)$, and $(w-1)^{d_{G-A'}(e,e')} \mid P_{G-A',e}(q, w) - P_{G-A,e}(q, w)$. Since $d_G(e, A) \leq d_G(e, A')$ and $d_G(e, A) \leq d_G(e, e') \leq d_{G-A'}(e, e')$, we have $(w - 1)^{d_G(e,A)} \mid P_{G,e}(q, w) - P_{G-A,e}(q, w)$. □

Lemma 56. *Let $G = (V, E)$ be a graph, $e \in E$, and $A, B \subseteq E \backslash \{e\}$, then the Taylor series near $w = 1$ of $P_{G-A,e}(q, w)$ and $P_{G-B,e}(q, w)$ satisfies*

$$(w - 1)^{d_G(e_1,A \neq B)} \mid P_{G-A,e}(q, w) - P_{G-B,e}(q, w).$$

Proof. Let $G' = G - (A \cup B)$, $A' = A \backslash B$ and $B' = B \backslash A$, then

$$\begin{aligned} P_{G-A,e}(q, w) - P_{G-B,e}(q, w) =& P_{G'-A',e}(q, w) - P_{G'-B',e}(q, w) \\ =& [P_{G'-A',e}(q, w) - P_{G',e}(q, w)] + [P_{G',e}(q, w) - P_{G'-B',e}(q, w)]. \end{aligned}$$

By the previous lemma, we have $(w-1)^{d_{G'}(e,A')} \mid P_{G'-A',e}(q, w)-P_{G',e}(q, w)$ and $(w-1)^{d_{G'}(e,B')} \mid P_{G',e}(q, w)-P_{G'-B',e}(q, w)$. Since $d_G(e, A \neq B) = \min\{d_G(e, A'), d_G(e, B')\} \leq \min\{d_{G'}(e, A'), d_{G'}(e, B')\}$, we are done. □

Combining Lemmas 16, 53 and 56, we can establish the edge-type SSM result for the Potts model (Theorem 50).

Remark 57. *Ratio $1/P_{G,e}(w)$ also exhibits edge-type SSM. Since $1/P_{G,e}(w)$ still avoid 0 and 1 and the $0 < 1/P_{G,e}(0) = \frac{Z_G(q,0)}{Z_G(q,0)+Z_{G'}(q,0)+Z} < 1$, the uniform bound can be obtained by Lemma 6. Also, LDC can still be established by the same technique.*

5.4 Vertex-type SSM for Ising

Similar to the edge-type SSM of the Ising model in Theorem 13, we also establish a vertex-type SSM.

Theorem 58. *Fix edge activity $\beta \geq 1$ and uniform external $\lambda \in \mathbb{D}_{\frac{1}{\beta}}$ for Ising model, and $c \in [0, 1)$. Then there exist constant $C > 0$ and $r > 1$ such that for all graph $G = (V, E)$, $v \in V$, $A, B \subseteq V \backslash \{v\}$, let $m = \{\lambda_v \to c\lambda\}$, $m_1 = \{\lambda_u \to c\lambda\}_{u \in A}$, $m_2 = \{\lambda_u \to c\lambda\}_{u \in B}$, we have*

$$\left| P^{m_1}_{G,m} - P^{m_2}_{G,m} \right| \leq Cr^{-d_G(v,m_1 \neq m_2)}.$$

5.4.1 Vertex-type LDC

Lemma 59. *For $\boldsymbol{\beta} \geq 1$, $c \in [0,1)$, $G = (V,E)$ be a graph, $v \in V$, $A \subseteq V\backslash\{v\}$, $\boldsymbol{\lambda} \in \mathbb{D}^V$, $m = \{\lambda_v z \to c\lambda_v z\}$, $m_1 = \{\lambda_u z \to c\lambda_u z\}_{u\in A}$, then*

$$z^{d_G(v,A)+1} \mid P_{G,m}(\beta, \boldsymbol{\lambda} z) - P^{m_1}_{G,m}(\beta, \boldsymbol{\lambda} z).$$

Proof.

$$\begin{aligned}P_{G,m}(\beta, \boldsymbol{\lambda} z) - P^{m_1}_{G,m}(\beta, \boldsymbol{\lambda} z) =& \frac{Z^m_G(\beta, \boldsymbol{\lambda} z)}{Z_G(\beta, \boldsymbol{\lambda} z)} - \frac{Z^{m,m_1}_G(\beta, \boldsymbol{\lambda} z)}{Z^{m_1}_G(\beta, \boldsymbol{\lambda} z)} \\ =& \frac{Z^m_G(\beta, \boldsymbol{\lambda} z) Z^{m_1}_G(\beta, \boldsymbol{\lambda} z) - Z^{m,m_1}_G(\beta, \boldsymbol{\lambda} z) Z_G(\beta, \boldsymbol{\lambda} z)}{Z_G(\beta, \boldsymbol{\lambda} z) Z^{m_1}_G(\beta, \boldsymbol{\lambda} z)}.\end{aligned}$$

Clearly $\frac{1}{Z_G(\beta,\boldsymbol{\lambda} z) Z^{m_1}_G(\beta,\boldsymbol{\lambda} z)}$ is analytic near $z = 0$. Proof of Lemma 19 also apply to vertex, then we have $z^{d_G(v,A)+1} \mid P_{G,m}(\beta, \boldsymbol{\lambda} z) - P^{m_1}_{G,m}(\beta, \boldsymbol{\lambda} z)$. ☐

Lemma 60. *For $\beta > 1$, $c \in [0,1)$, $G = (V,E)$ be a graph, $\boldsymbol{\lambda} \in \mathbb{D}^V$, $v \in V$, $A, B \subseteq V\backslash\{v\}$, $m = \{\lambda_v z \to c\lambda_v z\}$, $m_1 = \{\lambda_u z \to c\lambda_u z\}_{u\in A}$, $m_2 = \{\lambda_u z \to c\lambda_u z\}_{u\in B}$, then*

$$z^{d_G(v,m_1 \neq m_2)+1} \mid P^{m_1}_{G,m}(\beta, \boldsymbol{\lambda} z) - P^{m_2}_{G,m}(\beta, \boldsymbol{\lambda} z)$$

where $m_1 \neq m_2$ is vertex set where m_1 and m_2 differ.

Proof. Consider $\boldsymbol{\lambda}' z$ as the uniform external field $\boldsymbol{\lambda} z$ applied $m_1 \cap m_2$, let $m_1' = m_1\backslash m_2$, $m_2' = m_2\backslash m_1$, then

$$\begin{aligned}P^{m_1}_{G,m}(\beta, \boldsymbol{\lambda} z) - P^{m_2}_{G,m}(\beta, \boldsymbol{\lambda} z) =& P^{m_1'}_{G,m}(\beta, \boldsymbol{\lambda}' z) - P^{m_2'}_{G,m}(\beta, \boldsymbol{\lambda}' z) \\ =& [P^{m_1'}_{G,m}(\beta, \boldsymbol{\lambda}' z) - P_{G,m}(\beta, \boldsymbol{\lambda}' z)] + [P_{G,m}(\beta, \boldsymbol{\lambda}' z) - P^{m_2'}_{G,m}(\beta, \boldsymbol{\lambda}' z)].\end{aligned}$$

By the previous lemma, we have $z^{d_G(v,m_1')+1} \mid P_{G,m_1'}(\beta, \boldsymbol{\lambda}' z) - P_{G,m}(\beta, \boldsymbol{\lambda}' z)$ and $z^{d_G(v,m_2')+1} \mid P_{G,m}(\beta, \boldsymbol{\lambda}' z) - P_{G,m_2'}(\beta, \boldsymbol{\lambda}' z)$. Since $d_G(v, m_1 \neq m_2) = \min\{d_G(v, m_1'), d_G(v, m_2')\}$, we are done. ☐

5.4.2 Uniform bound of vertex-type ratio

Lemma 61 ([SY24, Corollary 40])**.** *Let G be a graph and v be a vertex in G. Then the partition function of Ising model $Z^+_{G,v}(\beta, \boldsymbol{\lambda})$ can be expressed as:*

$$Z^+_{G,v}(\beta, \boldsymbol{\lambda}) = \lambda_v Z_{G\backslash\{v\}}(\beta, \boldsymbol{\lambda}^{v^+})$$

where $Z_{G\backslash\{v\}}(\beta, \boldsymbol{\lambda}^{v^+})$ is the partition function of the Ising model with non-uniform external fields $\boldsymbol{\lambda}^{v^+}$ on the graph $G\backslash\{v\}$ obtained from G by deleting v, and $\lambda^{v^+}_w = \lambda_w$ for $w \in V\backslash(N(v) \cup \{v\})$ and $\lambda^{v^+}_w = \beta\lambda_w$ for $w \in N(v)$.

Lemma 62. *Let $G = (V, E)$ be a graph, $\beta > 1$, $\boldsymbol{\lambda} \in \mathbb{D}^V_{\frac{1}{\beta}}$, $v \in V(G)$, if $\lambda' \in \mathbb{D}_{\frac{1}{\beta}}$ and $\lambda'_v \neq \lambda_v$, then $P_{G,\{\lambda_v \to \lambda'\}}(\beta, \boldsymbol{\lambda})$ avoid 0 and 1.*

Proof. By Lee–Yang theorem, it is trivial that $P_{G,\{\lambda_v \to \lambda'\}}(\beta, \boldsymbol{\lambda}) \neq 0$. We prove the ratio avoids 1.

$$
\begin{aligned}
&Z_G(\beta,\boldsymbol{\lambda}) - Z_G(\beta,\boldsymbol{\lambda}') \\
=&Z^+_{G,v}(\beta,\boldsymbol{\lambda}) + Z^-_{G,v}(\beta,\boldsymbol{\lambda}) - Z^+_{G,v}(\beta,\boldsymbol{\lambda}') - Z^-_{G,v}(\beta,\boldsymbol{\lambda}') \\
=&Z^+_{G,v}(\beta,\boldsymbol{\lambda}) - Z^+_{G,v}(\beta,\boldsymbol{\lambda}') \\
=&(\lambda_v - \lambda'_v) Z_{G\backslash\{v\}}(\beta,\boldsymbol{\lambda}^{v^+}) \quad \text{(see Lemma 61)}
\end{aligned}
$$

Since $\boldsymbol{\lambda} \in \mathbb{D}^V_{\frac{1}{\beta}}$, then $\boldsymbol{\lambda}^{v^+} \in \mathbb{D}^{V\backslash\{v\}}$, by Lee–Yang theorem, $Z_{G\backslash\{v\}}(\beta,\boldsymbol{\lambda}^{v^+}) \neq 0$, thus the ratio avoid 1.

□

Lemma 63. *Fix $\beta \geq 1$ and $c \in [0,1)$, then for any compact set $S \subseteq \mathbb{D}_{\frac{1}{\beta}}\backslash\{0\}$, there exists a constant C such that for any graph $G = (V,E)$, vertex $v \in V$, $A \subseteq V\backslash\{v\}$, $m = \{\lambda_v \to c\lambda_v\}$, $m_1 = \{\lambda_u \to c\lambda_u\}_{u\in A}$, such that $|P^{m_1}_{G,m}(\beta,\lambda)| \leq C$ for all $\lambda \in S$.*

Proof. By Lemma 62, $P^{m_1}_{G,m}(\beta,\lambda)$ avoid 0 and 1 for all $\lambda \in \mathbb{D}_{\frac{1}{\beta}}\backslash\{0\}$. Pick a positive constant $\lambda' \in (0, \frac{1}{\beta})$, then $0 < P^{m_1}_{G,m}(\beta,\lambda') < 1$ always holds. Then by Lemma 6, the upper bound is got. □

Combining Lemmas 16, 60 and 63, we can establish the vertex type SSM.

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