# Strong Spatial Mixing for Ferromagnetic Ising Models: A Novel Perspective from Edge Activity

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**Abstract**

We prove a totally novel form of strong spatial mixing (SSM) for the ferromagnetic Ising model in terms of edge activities. This SSM property holds for the entire zero-free region of Lee–Yang Theorem, i.e., $\beta > 1$ and $|\lambda| < 1$ or symmetrically $|\lambda| > 1$. We also prove a form of SSM in terms of external fields in a smaller region $\beta > 1$ and $|\lambda| < 1/\beta$ or symmetrically $|\lambda| > \beta$. These SSM properties can be exploited to devise FPTASes via Weitz's and Barvinok's algorithms. Our proof is based on the framework introduced by [Reg23], namely local dependence of coefficients (LDC) and a uniform bound implies SSM. In order to establish LDC, we prove a division relation for the 2-spin system on general graphs.

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## 1 Introduction

Spin systems originated from statistical physics to model interactions between neighbors on graphs. In this paper, we focus on 2-state spin (2-spin) systems. Such a system is specified by two edge interaction parameters $\beta$ and $\gamma$, and a uniform external field $\lambda$. An instance is a graph $G = (V, E)$. A configuration $\sigma$ is a mapping $\sigma : V \to \{+, -\}$ which assigns one of the two spins $+$ and $-$ to each vertex in $V$. The weight $w(\sigma)$ of a configuration $\sigma$ is given by

$$w(\sigma) = \beta^{m_+(\sigma)}\gamma^{m_-(\sigma)}\lambda^{n_+(\sigma)},$$

where $m_+(\sigma)$ denotes the number of $(+, +)$ edges under the configuration $\sigma$, $m_-(\sigma)$ denotes the number of $(-, -)$ edges, and $n_+(\sigma)$ denotes the number of vertices assigned to spin $+$. The partition function $Z_G(\beta, \gamma, \lambda)$ of the system parameterized by $(\beta, \gamma, \lambda)$ is defined to be the sum of weights over all configurations, i.e.,

$$Z_G(\beta, \gamma, \lambda) = \sum_{\sigma:V\to\{+,-\}} w(\sigma).$$

We can also define the 2-spin system with non-uniform edge activities $\boldsymbol{\beta} = (\beta_e)_{e\in E}$, $\boldsymbol{\gamma} = (\gamma_e)_{e\in E}$ and external fields $\boldsymbol{\lambda} = (\lambda_v)_{v\in V}$. Define $E^+$ as the set of edges with both endpoints having spin $+$, $E^-$ as the set of edges with both endpoints having spin $-$, and $V^+$ as the set of vertices with spin $+$. Then, the partition function of the 2-spin system with non-uniform edge activities $\boldsymbol{\beta} = (\beta_e)_{e\in E}$, $\boldsymbol{\gamma} = (\gamma_e)_{e\in E}$ and external fields $\boldsymbol{\lambda} = (\lambda_v)_{v\in V}$ is

$$Z_G(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \sum_{\sigma:V\to\{+,-\}} \prod_{e\in E^+} \beta_e \prod_{e\in E^-} \gamma_e \prod_{v\in V^+} \lambda_v,$$

where we use $x_i$ notation to denote the $i$-index component of the vector $\boldsymbol{x}$.

Computing the partition function of the 2-spin system given an input graph $G$ is a very basic counting problem, and it is known to be #P-hard for all complex valued parameters $(\beta, \gamma, \lambda)$ but a few very restricted settings such as $\beta\gamma = 1$ or $\lambda = 0$ [Bar82, CCL13, CLX14]. Many natural combinatorial problems can be formulated as computing the partition function of the 2-spin system. For example, when $\beta = 0$ and $\gamma = 1$, $Z_G(0, 1, \lambda)$ is the independence polynomial of the graph $G$ (also known as the *hard-core model* in statistical physics); it counts the number of independent sets of the graph $G$ when $\lambda = 1$. When $\beta = \gamma$, such a 2-spin system is the famous *Ising model*.

In classical statistical mechanics the parameters $(\beta, \gamma, \lambda)$ are usually non-negative real numbers and $(\beta, \gamma) \neq (0, 0)$. Such 2-spin systems are divided into the *ferromagnetic* case ($\beta\gamma > 1$) and the *antiferromagnetic* case ($\beta\gamma < 1$). For non-negative $(\beta, \gamma, \lambda)$ that are not all zeros, the partition function can be viewed as the normalizing factor of the Gibbs distribution, which is the distribution where a configuration $\sigma$ is drawn with probability $\Pr_{G;\beta,\gamma,\lambda}(\sigma) = \frac{w(\sigma)}{Z_G(\beta,\gamma,\lambda)}$. However, it is meaningful to consider parameters of complex values. First, the parameters are generally complex valued for quantum computation. For instance, the partition function of 2-spin systems with complex parameters is closely related to the output probability amplitudes of quantum circuits [DDVM11, ICBB14, MB19]. Moreover, even in classical theory, the study of the location of *complex* zeros of the partition function $Z_G(\beta, \gamma, \lambda)$ connects closely to the analyticity of the free energy $\log Z_G(\beta, \gamma, \lambda)$, which is a classical notion in statistical physics for defining and understanding the phenomenon of *phase transitions*. One of the first and also the best known results regarding the zeros of the partition function is the Lee–Yang theorem [YL52, LY52] for the Ising model. This result was later extended to more general models by several people [Asa70, Rue71, GS73, New74, LS81].

Another standard notion for formalizing phase transitions in the 2-spin system is *correlation decay*, which refers to that correlations between spins decay exponentially with the distance between them.

The two notions of phase transitions can also be exploited directly to devise fully polynomial-time deterministic approximation schemes (FPTAS) for computing the partition function of the 2-spin system. The method associated with correlation decay, or more precisely *strong spatial mixing* (SSM) was originally developed by Weitz [Wei06] for the hard-core model. It turns out to be a very powerful tool for antiferromagnetic 2-spin systems [ZLB11, LLY12, LLY13, SST14]. While for ferromagnetic 2-spin systems, limited results [ZLB11, GL18] have been obtained via the correlation decay method. The method turning complex zero-free regions of the partition function into FPTASes was developed by Barvinok [Bar16], and extended by Patel and Regts [PR17]. It is usually called the *Taylor polynomial interpolation* method. Motivated by this method, several complex zero-free regions have been obtained for hard-core models [PR19, BCSV23], Ising models [LSS19a, PR20], and general 2-spin systems [GLL20, SS21].

In this paper, we prove a new form of SSM for the ferromagnetic Ising model in terms of edge activities. This SSM property holds for the entire zero-free region of Lee–Yang Theorem, i.e., $\beta > 1$ and $|\lambda| < 1$ or symmetrically $|\lambda| > 1$. We also prove a form of SSM in terms of external fields in a smaller region $|\lambda| < 1/\beta$ or symmetrically $|\lambda| > \beta$. These SSM properties can be exploited to devise FPTASes via Weitz's and Barvinok's algorithms. Our proof is based on the framework introduced by [Reg23] and developed by [SY24], namely local dependence of coefficients (LDC) and a uniform bound implies SSM. In order to establish LDC, we prove a division relation for the 2-spin system on general graphs which is our main technical contribution. Such a division relation to some degree can be viewed as a generalization of the Christoffel–Darboux type identity for the 2-spin system on trees [SY24], but it is proved in an entirely different way.

The paper is organized as follows. In Section 2, we introduce the framework of zero-freeness implying SSM. In Section 3, we prove a division relation for the 2-spin system on general graphs.

In Section 4 , we introduce and prove the SSM property in terms of edge activities for the ferromagnetic Ising model using Lee–Yang theorem and the division relation. In Section 5 , we discuss how this edge-type SSM gives an FPTAS. We introduce and prove the SSM property in terms of external fields similarly in the appendix.

# 2 Preliminaries

## 2.1 Notation and definitions

Given a graph $G = (V, E)$, in 2-spin systems, the marginal probability that a vertex $v$ is assigned to spin $(+)$ is denoted by $P_{G,v} = \frac{Z^{+}_{G,v}(\beta,\gamma,\lambda)}{Z_G(\beta,\gamma,\lambda)}$, where $Z^{+}_{G,v}(\beta, \gamma, \lambda)$ is the contribution to $Z_G(\beta, \gamma, \lambda)$ over all configurations that assign spin $(+)$ to vertex $v$. Let $\sigma_\Lambda$ be a partial configuration of some vertices $\Lambda \subset V(G)$. The marginal probability that a vertex $v \notin \Lambda$ is assigned to spin $(+)$ conditioned on the partial configuration $\sigma_\Lambda$ is denoted by $P^{\sigma_\Lambda}_{G,v} = \frac{Z^{\sigma_\Lambda}_{G,v}(\beta,\gamma,\lambda)}{Z^{\sigma_\Lambda}_{G}(\beta,\gamma,\lambda)}$, where $Z^{\sigma_\Lambda}_{G}(\beta, \gamma, \lambda)$ is the contribution to $Z_G(\beta, \gamma, \lambda)$ over all configurations that agree with $\sigma_\Lambda$ on $\Lambda$ and $Z^{\sigma_\Lambda}_{G,v}(\beta, \gamma, \lambda)$ is the contribution to $Z^{\sigma_\Lambda}_{G}(\beta, \gamma, \lambda)$ that assign spin $(+)$ to vertex $v$.

Such a marginal probability can be viewed as a ratio of the partition function. Thus we can define a new type of ratio under the vector value setting, where we change the value of the vector value $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ instead of pinning vertices. For example, we can see that the operation of pinning a vertex $v$ to spin $(-)$ is equivalent to setting $\lambda_v = 0$. Also, the operation of removing an edge $e$ is equivalent to setting $\beta_e = \gamma_e = 1$.

The ratio of the partition function after changing the vector value from $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ to $(\boldsymbol{\beta}', \boldsymbol{\gamma}', \boldsymbol{\lambda}')$ is

$$\frac{Z_G(\boldsymbol{\beta}', \boldsymbol{\gamma}', \boldsymbol{\lambda}')}{Z_G(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})}.$$

We can use a mapping $m$ to denote the change of the vector value, a mapping is a set of complex function $\mathbb{C} \to \mathbb{C}$ which maps a component of the vector value to a new value. For example, when we pin a vertex $v$ with $(-)$ spin, we set $\lambda_v = 0$, then $m = \{\lambda_v \to 0\}$. When we remove an edge $e$, we set $\beta_e = \gamma_e = 1$, then $m = \{\beta_e \to 1, \gamma_e \to 1\}$. We denote $Z_G^m(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ as the partition function of the system after the mapping $m$. Then we can define the ratio of the partition function after the mapping $m$ to the original partition function as

$$P_{G,m}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \frac{Z_G^m(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})}{Z_G(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})}.$$

Moreover, we can define the ratio of the partition function conditioning on a pre-described mapping, denoted by

$$P_{G,m}^{m_1}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \frac{Z_G^{m_1,m}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})}{Z_G^{m_1}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})}.$$

The ratio is well defined if $m$ and $m_1$ do not change the same component of the vector value. If no confusion, we will omit the vector value $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ in the notation.

Having such a new type of ratio, we can define the Strong Spatial Mixing (SSM) property on the vector value version. Firstly, we review the initial definition of SSM for the 2-spin system.

**Definition 2.1** (Strong spatial mixing)**.** *Fix complex parameters $\beta, \gamma, \lambda$ where $(\beta, \gamma) \neq (0, 0)$ and $\lambda \neq 0$, and a family of graphs $\mathcal{G}$. The corresponding 2-spin system defined on $\mathcal{G}$ with parameters $(\beta, \gamma, \lambda)$ is said to satisfy* strong spatial mixing (SSM) *with exponential rate $r > 1$ if there exists a constant $C$ such that for any $G = (V, E) \in \mathcal{G}$, any feasible partial configurations $\sigma_{\Lambda_1}$ and $\tau_{\Lambda_2}$ where $\Lambda_1$ may be different with $\Lambda_2$, and any vertex $v$ proper to $\sigma_{\Lambda_1}$ and $\tau_{\Lambda_2}$, we have*

$$\left| P_{G,v}^{\sigma_{\Lambda_1}} - P_{G,v}^{\tau_{\Lambda_2}} \right| \leq C r^{-d_G(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}.$$

*Here, we denote $\sigma_{\Lambda_1} \neq \tau_{\Lambda_2}$ the set $(\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1) \cup \{v \in \Lambda_1 \cap \Lambda_2 : \sigma_{\Lambda_1}(v) \neq \tau_{\Lambda_2}(v)\}$ (i.e., the set on which $\sigma_{\Lambda_1}$ and $\tau_{\Lambda_2}$ differ with each other), and $d_G(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})$ is the shortest distance from $v$ to any vertex in $\sigma_{\Lambda_1} \neq \tau_{\Lambda_2}$.*

Replace the partial configuration with the mapping, we can similarly define the SSM property on the vector value version. Since our result focus on the Ising model, for simplicity, we directly give our result on the Ising model and from now on we will only consider the Ising model.

For $\rho > 0$, we define the open disk $\mathbb{D}_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ and write $\mathbb{D}$ as $\mathbb{D}_1$.

**Theorem 2.2** (edge-type SSM)**.** *Fix uniform external field $\lambda \in \mathbb{D}$ and two constants $0 < C_1 < C_2$ , then there exist constants $C > 0$ and $r > 1$ such that for all pair of graph $G = (V, E)$ and vector edge activity $\boldsymbol{\beta} \geq 1$, $e \in E$, $A, B \subset E \backslash \{e\}$, let $m = \{\beta_e \to \beta'_e\}$, $m_1 = \{\beta_f \to \beta_f^A\}_{f \in A}$,*

$m_2 = \{\beta_f \to \beta_f^B\}_{f\in B}$ *where* $\beta_e' \geq 1$, $\beta_f^A \geq 1$ *for* $f \in A$, $\beta_f^B \geq 1$ *for* $f \in B$ *and* $C_1 \leq \frac{\beta_e'}{\beta_e} \leq C_2$, *we have*

$$\left| P_{G,m}^{m_1} - P_{G,m}^{m_2} \right| \leq C r^{-d_G(e, m_1 \neq m_2)}.$$

*Here, we denote* $m_1 \neq m_2 = (A\backslash B) \cup (B\backslash A) \cup \{f \in A \cap B : \beta_f^A \neq \beta_f^B\}$, *the set of edges on which* $m_1$ *and* $m_2$ *differ with each other, and* $d_G(e, m_1 \neq m_2)$ *is the shortest distance from any end of* $e$ *to any end of edge in* $m_1 \neq m_2$.

As a corollary, the ratio that only removes the edge defined as $P_{G,e}(\beta, \lambda) = \frac{Z_{G\backslash e}(\beta,\lambda)}{Z_G(\beta,\lambda)}$ exhibits SSM.

**Corollary 2.3.** *Fix* $\beta \geq 1$ *and uniform external filed* $\lambda \in \mathbb{D}$ *for Ising model, then there exist constants* $C > 0$ *and* $r > 1$ *such that for all graph* $G = (V, E)$, $e \in E$, *edge sets* $A, B \subset E\backslash\{e\}$, *we have*

$$\left| P_{G\backslash A,e} - P_{G\backslash B,e} \right| \leq C r^{d_G(e, A\neq B)}$$

*where* $A \neq B = (A\backslash B) \cup (B\backslash A)$, *the symmetric difference of* $A$ *and* $B$, *and* $d_G(e, A \neq B)$ *is the shortest distance from* $e$ *to any edge in* $A \neq B$.

## 2.2 LDC amd uniform bound implies SSM

The framework that LDC and uniform bound implies SSM was originally introduced in [Reg23]. Here, we adopt the definition of LDC from [SY24]. For two complex functions $f(\lambda)$ and $g(\lambda)$ analytic near 0, we denote by $\lambda^k \mid f(\lambda) - g(\lambda)$ the property that their Taylor series $f(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i$ and $g(\lambda) = \sum_{i=0}^{\infty} b_i \lambda^i$ near $\lambda = 0$ satisfy $a_i = b_i$ for $0 \leq i \leq k-1$. A region is a connected open set in $\mathbb{C}$, especially, an open disk with an inner point removed is also a region.

**Definition 2.4** (LDC)**.** *For all pair of graph* $G = (V, E)$ *and vector edge activity* $\boldsymbol{\beta} \geq 1$, $e \in E$, $A, B \subset E\backslash\{e\}$, *let* $m = \{\beta_e \to \beta_e'\}$, $m_1 = \{\beta_f \to \beta_f^A\}_{f\in A}$, $m_2 = \{\beta_f \to \beta_f^B\}_{f\in B}$ *where* $\beta_e' \geq 1$, $\beta_f^A \geq 1$ *for* $f \in A$, $\beta_f^B \geq 1$ *for* $f \in B$ *and* $C_1 \leq \frac{\beta_e'}{\beta_e} \leq C_2$, *then*

$$\lambda^{d_G(e, m_1\neq m_2)+1} \mid P_{G,m}^{m_1}(\boldsymbol{\beta}, \lambda) - P_{G,m}^{m_2}(\boldsymbol{\beta}, \lambda)$$

Similarly, one can define vertex-type LDC for vertex-type ratio function.

**Remark 2.5.** *By Lee–Yang theorem, if* $\boldsymbol{\beta} \geq 1$ *then the partition function of the Ising model* $Z_G(\boldsymbol{\beta}, \boldsymbol{\lambda}) \neq 0$ *for all* $\boldsymbol{\lambda} \in \mathbb{D}^V$. *Thus the ratio* $P_{G,m}^{m_1}(\boldsymbol{\beta}, \lambda)$ *in definition 2.4 is always analytic on* $\lambda \in \mathbb{D}$. *In order to prove the LDC property, we only need to focus on the Taylor series of the ratio near* $\lambda = 0$.

Exactly following the proofs in [SY24], we have the following lemma.

**Lemma 2.6.** *Fix* $0 < C_1 < C_2$, $U \subset \mathbb{D}$ *is a complex region containing* 0 *and* $\partial U \subset \mathbb{D}$. *If there exist a constant* $C > 0$ *such that for all pair of graph* $G = (V, E)$ *and vector edge activity* $\boldsymbol{\beta} \geq 1$, $P_{G,m}^{m_1}(\boldsymbol{\beta}, \lambda) \leq C$ *holds on* $\lambda \in \partial U$ *and the LDC property holds, then the edge-type ratio function satisfies edge-type SSM for any* $z \in U$.

Similarly, vertex-type LDC and a uniform bound for vertex-type ratio will imply vertex-type SSM.

In [Reg23], Regts introduce Montel's theorem to get a uniform bound for a family of analytic functions. In particular, the following lemma is proved.

**Lemma 2.7.** *Let $U$ be a complex region and $\mathcal{F}$ be a family of holomorphic functions $f : U \to \mathbb{C}$ such that $f(U) \subset \mathbb{C}\backslash\{0,1\}$ for all $f \in \mathcal{F}$. If there exists $z_0 \in U$ and $C > 0$ such that $|f(z_0)| \le C$ for all $f \in \mathcal{F}$. Then for any compact subset $S \subset U$, there exists a positive constant $C_1$ such that for all $f \in \mathcal{F}$ and $z \in S$, we have $|f(z)| \le C_1$.*

# 3 Division Relation

**Lemma 3.1.** *Let $G = (V, E)$ be a graph with 2-spin vector value parameters $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$, and $m_1, m_2$ be vector value mapping of $G$. Assuming that $m_1$ and $m_2$ only change the value in subgraph $G_1$ and $G_2$ respectively and $G_1 \cap G_2 = \varnothing$. If for all $v \in V$ the external field $\lambda_v$ is dividable by $\lambda$ and all changed external field by $m_1, m_2$ is also dividable by $\lambda$, then*

$$\lambda^{d_G(G_1,G_2)+1} \mid Z_G(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) Z_G^{m_1,m_2}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) - Z_G^{m_1}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) Z_G^{m_2}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$$

*where $d_G(G_1, G_2) = \min_{u \in G_1, v \in G_2} d_G(u, v)$.*

*Proof.* For simplicity, we omit $(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ in the notation. Let $\mathcal{S}_G = V \to \{+, -\}$, then

$$\begin{aligned}
& Z_G Z_G^{m_1,m_2} - Z_G^{m_1} Z_G^{m_2} \\
= & \sum_{\sigma \in \mathcal{S}_G} w_G(\sigma) \sum_{\sigma \in \mathcal{S}_G} w_G^{m_1,m_2}(\sigma) - \sum_{\sigma \in \mathcal{S}_G} w_G^{m_1}(\sigma) \sum_{\sigma \in \mathcal{S}_G} w_G^{m_2}(\sigma) \\
= & \sum_{\substack{(\sigma_1,\sigma_2) \in \\ (\mathcal{S}_G \times \mathcal{S}_G)}} w_G(\sigma_1) w_G^{m_1,m_2}(\sigma_2) - \sum_{\substack{(\sigma_3,\sigma_4) \in \\ (\mathcal{S}_G \times \mathcal{S}_G)}} w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4)
\end{aligned}$$

For $(\sigma, \tau) \in \mathcal{S}_G \times S_G$, we consider the or operator of spin defined by $or(-,-) = -$, $or(-,+) = +$, $or(+,-) = +$ and $or(+,+) = +$. The or operator of two configurations is denoted by $or(\sigma, \tau) = \{v \in G : or(\sigma(v), \tau(v))\}$.

If there exists a $(+)$ path (all vertices in the path have $(+)$ spin) in $or(\sigma, \tau)$ connecting some components of $G_1$ and $G_2$ in $G$, then the path has at least $d_G(G_1, G_2) + 1$ vertices with spin $(+)$. Since the external field $\lambda$ is dividable by $\lambda_v$ for all $v \in V$ and all changed external field by $m_1, m_2$ is also dividable by $\lambda$, the term $w_G(\sigma) w_G^{m_1,m_2}(\tau)$ and $w_G^{m_1}(\sigma) w_G^{m_2}(\tau)$ must have a factor $\lambda^{d_G(G_1,G_2)+1}$.

Thus we only need to consider the terms such that there are no $(+)$ paths connecting some components of $G_1$ and $G_2$ in $or(\sigma, \tau)$, we call such a pair configurations good. We will show that the sum of the remaining terms is exactly 0. To do this, we will show that there exists a bijection $f$ from good pair to good pair, such that for each good pair $(\sigma_1, \sigma_2)$, $f(\sigma_1, \sigma_2) = (\sigma_3, \sigma_4)$ is still a good pair and $w_G(\sigma_1) w_G^{m_1,m_2}(\sigma_2) = w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4)$.

Let $(\sigma_1, \sigma_2)$ be a good pair, consider the subgraph $G'$ of $G$ which is induced by $\{v \in V(G \backslash (G_1 \cup G_2)) | or(\sigma_1, \sigma_2)(v) = +\} \cup V(G_1) \cup V(G_2)$. By the definition of good pair, there are no $(+)$ paths connecting some components of $G_1$ and $G_2$ in $G'$.

Let $S$ be the minimal vertex set containing all connected components of $G'$ which intersect with $G_1$, i.e., $S = \bigcup_{A \text{ is a connected component of } G', A \cap G_1 \neq \varnothing} V(A)$, and $T$ be $V(G) \backslash S$ (easily to see $G_1 \subset G[S]$ and $G_2 \subset G[T]$).

Define function $f$ as follows:

$$f(\sigma_1, \sigma_2) = (\sigma_2|_{S(\sigma_1,\sigma_2)} \oplus \sigma_1|_{T(\sigma_1,\sigma_2)}, \sigma_1|_{S(\sigma_1,\sigma_2)} \oplus \sigma_2|_{T(\sigma_1,\sigma_2)})$$

where | denotes the restriction of the configuration on the vertex set, $\oplus$ denotes the union of two partial configurations. The function $f$ is well-defined since $S(\sigma_1, \sigma_2)$ and $T(\sigma_1, \sigma_2)$ are well defined by good pair $(\sigma_1, \sigma_2)$.

This mapping $f$ is just exchange and recombine the $S$ part of two configurations. Let $(\sigma_3, \sigma_4) = f(\sigma_1, \sigma_2)$, it's easy to see that $or(\sigma_1, \sigma_2) = or(\sigma_3, \sigma_4)$, thus $S(\sigma_1, \sigma_2) = S(\sigma_3, \sigma_4)$. Thus $f$ is a bijection and its inverse function is itself (swap twice is the original pair).

For each good pair $(\sigma_1, \sigma_2)$ and $(\sigma_3, \sigma_4) = f(\sigma_1, \sigma_2)$. Let $S = S(\sigma_1, \sigma_2) = S(\sigma_3, \sigma_4)$ and $T = T(\sigma_1, \sigma_2) = T(\sigma_3, \sigma_4)$. Let $s(e, \sigma)$ be the indicator function whether $e \in cut_G(S, T)$ under the configuration $\sigma$ is $(-, -)$, where $e = (u, v), u \in S, v \in T$. We will show that $s(e, \sigma_1) + s(e, \sigma_2) = s(e, \sigma_3) + s(e, \sigma_4)$. From the definition of $S$ and $T$, there are no $(+, +)$ edges between $S$ and $T$ under the configurations $\sigma' = or(\sigma_1, \sigma_2) = or(\sigma_3, \sigma_4)$, thus at least one of $\sigma'(u)$ and $\sigma'(v)$ is $(-)$. W.l.o.g, we assume $\sigma'(u) = (-)$, then $\sigma_1(u) = \sigma_2(u) = \sigma_3(u) = \sigma_4(u) = (-)$. Thus $s(e, \sigma_1) + s(e, \sigma_2)$ is the number of $(-)$ spins between $\sigma_1(v)$ and $\sigma_2(v)$ and $s(e, \sigma_3) + s(e, \sigma_4)$ is the number of $(-)$ spins between $\sigma_3(v)$ and $\sigma_4(v)$. Since $\sigma_1(v) = \sigma_3(v)$ and $\sigma_2(v) = \sigma_4(v)$, we have $s(e, \sigma_1) + s(e, \sigma_2) = s(e, \sigma_3) + s(e, \sigma_4)$.

Write $C = cut_G(S, T)$, by the definition of $w$, we have

$$
\begin{aligned}
& w_G(\sigma_1) w_G^{m_1, m_2}(\sigma_2) \\
=& \prod_{e \in C} \beta_e^{s(e, \sigma_1)} w_{G[S]}(\sigma_1|_S) w_{G[T]}(\sigma_1|_T) \prod_{e \in C} \beta_e^{s(e, \sigma_2)} w_{G[S]}^{m_1}(\sigma_2|_S) w_{G[T]}^{m_2}(\sigma_2|_T) \\
=& \prod_{e \in C} \beta_e^{s(e, \sigma_1) + s(e, \sigma_2)} w_{G[S]}^{m_1}(\sigma_2|_S) w_{G[T]}(\sigma_1|_T) w_{G[S]}(\sigma_1|_S) w_{G[T]}^{m_2}(\sigma_2|_T) \\
=& \prod_{e \in C} \beta_e^{s(e, \sigma_3) + s(e, \sigma_4)} w_{G[S]}^{m_1}(\sigma_3|_S) w_{G[T]}(\sigma_3|_T) w_{G[S]}(\sigma_4|_S) w_{G[T]}^{m_2}(\sigma_4|_T) \\
=& \prod_{e \in C} \beta_e^{s(e, \sigma_3)} w_{G[S]}^{m_1}(\sigma_3|_S) w_{G[T]}(\sigma_3|_T) \prod_{e \in C} \beta_e^{s(e, \sigma_4)} w_{G[S]}(\sigma_4|_S) w_{G[T]}^{m_2}(\sigma_4|_T) \\
=& w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4)
\end{aligned}
$$

Thus the sum of the remaining terms is exactly 0, which is dividable by $\lambda^{d_G(G_1, G_2)+1}$. $\square$

As applications of the above property, we can prove new SSM results for on Ising model in the following sections, where we see the partition function as a univariate polynomial in $\lambda$.

# 4 Edge-type SSM

## 4.1 Edge-type LDC

**Lemma 4.1.** *Let $G = (V, E)$ be a graph with edge activities $\boldsymbol{\beta} \geq 1$ and uniform external field $\lambda \in \mathbb{D}$. Let $e \in G$ and $A \subset E \backslash \{e\}$, $m = \{\beta_e \to \beta'_e\}$ with $\beta'_e \geq 1$ and $m_1 = \{\beta_{e'} \to \beta'_{e'} | e' \in A\}$ where $\beta'_{e'} \geq 1$ for all $e' \in A$. Then the Taylor series of $P_{G,m}(\boldsymbol{\beta}, \lambda)$ and $P_{G,m}^{m_1}(\boldsymbol{\beta}, \lambda)$ near $\lambda = 0$ satisfy*

$$\lambda^{d_G(e,A)+1} \mid P_{G,m}(\boldsymbol{\beta}, \lambda) - P_{G,m}^{m_1}(\boldsymbol{\beta}, \lambda)$$

*Proof.*

$$\begin{aligned}P_{G,m}(\boldsymbol{\beta},\lambda)-P^{m_1}_{G,m}(\boldsymbol{\beta},\lambda)=&\frac{Z^m_G(\boldsymbol{\beta},\lambda)}{Z_G(\boldsymbol{\beta},\lambda)}-\frac{Z^{m,m_1}_G(\boldsymbol{\beta},\lambda)}{Z^{m_1}_G(\boldsymbol{\beta},\lambda)}\\=&\frac{Z^m_G(\boldsymbol{\beta},\lambda)Z^{m_1}_G(\boldsymbol{\beta},\lambda)-Z^{m,m_1}_G(\boldsymbol{\beta},\lambda)Z_G(\boldsymbol{\beta},\lambda)}{Z_G(\boldsymbol{\beta},\lambda)Z^{m_1}_G(\boldsymbol{\beta},\lambda)}.\end{aligned}$$

Clearly $\frac{1}{Z_G(\boldsymbol{\beta},\lambda)Z^{m_1}_G(\boldsymbol{\beta},\lambda)}$ is analytic near 0. Combining with Lemma 3.1, we have $\lambda^{d_G(e,A)+1} \mid P_{G,m}(\boldsymbol{\beta},\lambda)-P^{m_1}_{G,m}(\boldsymbol{\beta},\lambda)$. ☐

**Lemma 4.2.** *Let $G=(V,E)$ be a graph with edge activities $\boldsymbol{\beta}\geq 1$ and uniform external field $\lambda\in\mathbb{D}$. Let $e\in E$ and $A,B\subset E\backslash\{e\}$, $m=\{\beta_e\to\beta'_e\}$ with $\beta'_e\geq 1$ and $m_1=\{\beta_{e'}\to\beta^A_{e'}|e'\in A\}$ , $m_2=\{\beta_{e'}\to\beta^B_{e'}|e'\in B\}$ where $\beta^A_{e'}\geq 1$ for all $e'\in A$ and $\beta^B_{e'}\geq 1$ for all $e'\in B$. Then the Taylor series of $P^{m_1}_{G,m}(\boldsymbol{\beta},\lambda)$ and $P^{m_2}_{G,m}(\boldsymbol{\beta},\lambda)$ near $\lambda=0$ satisfy*

$$\lambda^{d_G(e,m_1\neq m_2)+1}\mid P^{m_1}_{G,m}(\boldsymbol{\beta},\lambda)-P^{m_2}_{G,m}(\boldsymbol{\beta},\lambda).$$

*Proof.* Consider $\boldsymbol{\beta}'$ is $\boldsymbol{\beta}$ changed by $m_1\cap m_2$, $m'_1=m_1\backslash m_2$, $m'_2=m_2\backslash m_1$, then

$$\begin{aligned}P^{m_1}_{G,m}(\boldsymbol{\beta},\lambda)-P^{m_2}_{G,m}(\boldsymbol{\beta},\lambda)=&P^{m'_1}_{G,m}(\boldsymbol{\beta}',\lambda)-P^{m'_2}_{G,m}(\boldsymbol{\beta}',\lambda)\\=&[P^{m'_1}_{G,m}(\boldsymbol{\beta}',\lambda)-P_{G,m}(\boldsymbol{\beta}',\lambda)]+[P_{G,m}(\boldsymbol{\beta}',\lambda)-P^{m'_2}_{G,m}(\boldsymbol{\beta}',\lambda)].\end{aligned}$$

By the previous lemma, we have $\lambda^{d_G(e,m'_1)+1}\mid P^{m'_1}_{G,m}(\boldsymbol{\beta}',\lambda)-P_{G,m}(\boldsymbol{\beta}',\lambda)$ and $\lambda^{d_G(e,m'_2)+1}\mid P_{G,m}(\boldsymbol{\beta}',\lambda)-P^{m'_2}_{G,m}(\boldsymbol{\beta}',\lambda)$. Since $d_G(e,m_1\neq m_b)=\min\{d_G(e,m'_1),d_G(e,m'_2)\}$, we are done. ☐

### 4.2 Uniform bound of edge type ratio

The celebrated Lee–Yang theorem states that if all vertex is in the unit disk, then the partition function of ferromagnetic Ising model is zero-free.

**Theorem 4.3** (Lee–Yang theorem)**.** *Let $G=(V,E)$ be a graph, and $\boldsymbol{\beta}=(\beta_e)_{e\in E}$ where $\beta_e\geq 1$ for all $e\in E$ and $\boldsymbol{\lambda}=(\lambda_v)_{v\in V}\in\mathbb{D}^{|V|}$ be the external field. Then the partition function of Ising model $Z_G(\boldsymbol{\beta},\boldsymbol{\lambda})\neq 0$.*

We are ready to prove the edge type ratio avoid 0 and 1.

**Lemma 4.4.** *Let $G=(V,E)$ be a graph, with edge activities $\boldsymbol{\beta}\geq 1$ and non-uniform external fields $\boldsymbol{\lambda}\in\mathbb{D}^V$, $e\in E$, if $\beta'\geq 1$ and $\beta'\neq\beta_e$, then $P_{G,\{\beta_e\to\beta'\}}(\boldsymbol{\beta},\boldsymbol{\lambda})$ avoid 0 and 1.*

*Proof.* Since $\beta'\geq 1$, by Lee–Yang theorem, it is trivial that $P_{G,\{\beta_e\to\beta'\}}(\boldsymbol{\beta},\boldsymbol{\lambda})\neq 0$. We prove the ratio avoid 1.

Let $e=(u,v)$, we have

$$
\begin{aligned}
& Z_G(\boldsymbol{\beta},\boldsymbol{\lambda}) - Z_G^{\{\beta_e\to\beta'\}}(\boldsymbol{\beta},\boldsymbol{\lambda}) \\
=& Z^{+,+}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) + Z^{-,-}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) + Z^{+,-}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) + Z^{-,+}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) \\
& - \frac{\beta'}{\beta_e} Z^{+,+}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) - \frac{\beta'}{\beta_e} Z^{-,-}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) - Z^{+,-}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) - Z^{-,+}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) \\
=& \frac{\beta_e-\beta'}{\beta_e}(Z^{+,+}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) + Z^{-,-}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}))
\end{aligned}
$$

Merge $u,v$ into a single vertex $w$ we get graph $G'$, set $\lambda_w=\lambda_u\lambda_v$, if parallel edges exist (i.e. $(u,x)\in E,(v,x)\in E$ for some $x\in V$), we merge them into a single edge and set $\beta_{(w,x)}=\beta_{(u,x)}\beta_{(v,x)}$. Write the partition function of $G'$ as $Z_{G'}(\boldsymbol{\beta}',\boldsymbol{\lambda}')$.

We can see $Z_{G'}(\boldsymbol{\beta}',\boldsymbol{\lambda}') = Z^{+}_{G',w}(\boldsymbol{\beta}',\boldsymbol{\lambda}') + Z^{-}_{G',w}(\boldsymbol{\beta}',\boldsymbol{\lambda}') = \frac{1}{\beta_e}(Z^{+,+}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}) + Z^{-,-}_{G,u,v}(\boldsymbol{\beta},\boldsymbol{\lambda}))$. Since $\boldsymbol{\lambda}'\in\mathbb{D}^{|V(G)-1|}$ and $\boldsymbol{\beta}'\geq 1$, by Lee–Yang theorem, $Z_G(\boldsymbol{\beta},\boldsymbol{\lambda})-Z_G^{\{\beta_e\to\beta'\}}(\boldsymbol{\beta},\boldsymbol{\lambda}) = (\beta_e-\beta')Z_{G'}(\boldsymbol{\beta}',\boldsymbol{\lambda}')\neq 0$. Thus the ratio avoid 1.

□

**Lemma 4.5** (uniform bound)**.** *Fix $0<C_1\leq C_2$ be two constant numbers, $S$ be a compact subset of $\mathbb{D}$. There exists a constant $C>0$ such that for any graph $G=(V,E)$ with any edge activities $\boldsymbol{\beta}\geq 1$, any $e\in E$, any $\beta'\geq 1$ with $C_1\leq\frac{\beta'}{\beta_e}\leq C_2$ and any $\lambda\in S$, we have $|P_{G,\{\beta_e\to\beta'\}}(\boldsymbol{\beta},\lambda)|\leq C$.*

*Proof.* We see the ratios as a family of functions of $\lambda$. It's trivial when $\beta'=\beta_e$, the ratio is exactly 1. So we only consider the family of ratio functions when $\beta'\neq\beta_e$. By Lemma 4.4, $P_{G,\{\beta_e\to\beta'\}}(\boldsymbol{\beta},\lambda)$ avoid 0 and 1 for all $\lambda\in\mathbb{D}$. Since $P_{G,\{\beta_e\to\beta'\}}(\boldsymbol{\beta},0)=\frac{\beta'}{\beta_e}\in[C_1,C_2]$ is bounded, by Lemma 2.7, the upper bound is got. □

Combining Lemmas 2.6, 4.2 and 4.5, we can establish the edge type SSM.

**Theorem 2.2** (edge-type SSM)**.** *Fix uniform external field $\lambda\in\mathbb{D}$ and two constants $0<C_1<C_2$ , then there exist constants $C>0$ and $r>1$ such that for all pair of graph $G=(V,E)$ and vector edge activity $\boldsymbol{\beta}\geq 1$, $e\in E$, $A,B\subset E\backslash\{e\}$, let $m=\{\beta_e\to\beta'_e\}$, $m_1=\{\beta_f\to\beta_f^A\}_{f\in A}$, $m_2=\{\beta_f\to\beta_f^B\}_{f\in B}$ where $\beta'_e\geq 1$, $\beta_f^A\geq 1$ for $f\in A$, $\beta_f^B\geq 1$ for $f\in B$ and $C_1\leq\frac{\beta'_e}{\beta_e}\leq C_2$, we have*

$$\left|P^{m_1}_{G,m}-P^{m_2}_{G,m}\right|\leq Cr^{-d_G(e,m_1\neq m_2)}.$$

*Here, we denote $m_1\neq m_2=(A\backslash B)\cup(B\backslash A)\cup\{f\in A\cap B:\beta_f^A\neq\beta_f^B\}$, the set of edges on which $m_1$ and $m_2$ differ with each other, and $d_G(e,m_1\neq m_2)$ is the shortest distance from any end of $e$ to any end of edge in $m_1\neq m_2$.*

# 5 FPTAS for Ising model

Fix $\beta>1$, $\Delta\in N^+$ and $\lambda\in\mathbb{D}$, we can design a new FPTAS for the partition function $Z_G(\beta,\lambda)$ of Ising model with uniform external field $\lambda$ using the above results. Even though this algorithm is definitely slower than Barvinok's algorithm.

Our algorithm is similar to both Weitz's and Barvinok's algorithms. In Weitz's algorithm, we pin a vertex at each step and compute an approximation of the ratio of the partition function via SSM, then multiply them to get the approximation of the partition function. In Barvinok's algorithm, we compute the first $O(\log(n))$ coefficients of the Taylor series of $\log(Z)$ and use the truncation of the Taylor series to get the approximation of the partition function.

In our algorithm, we delete an edge in each step, and compute the approximation of the ratio of partition function via computing the first $O(\log(n))$ coefficients of the Taylor series of the ratio of partition function. Then we multiply them to get the approximation of the partition function.

**Lemma 5.1.** *Fix $\beta > 1$, for all graph $G$ and edge $e \in G$, for any compact subset $S \subset \mathbb{D}$, there exists a constant $c > 0$ such that $|P_{G,e}(\beta, \lambda)| \geq c$ for all $\lambda \in S$.*

*Proof.* By Lemma 4.4, $1/P_{G,e}(\lambda)$ avoid 0 and 1. Since $1/P_{G,e}(0) = 1/\beta$ is bounded, by Lemma 2.7, the upper bound of $1/P_{G,e}(\lambda)$ is got. Thus the lower bound of $P_{G,e}(\lambda)$ is got. □

**Lemma 5.2** (c.f. Theorem 3.1 in [LSS19b])**.** *Fix $\Delta \in N^+$ and constant $C > 0$. There exist a deterministic* $\text{poly}(n/\varepsilon)$*-time algorithm that, given any $n$-vertex graph $G$ of maximum degree $\Delta$ and any $\varepsilon \in (0, 1)$, computes the first $C \log(n/\varepsilon)$ coefficients of the partition function of the Ising model on $G$ with edge activities $\beta$ and uniform external field $\lambda$.*

**Lemma 5.3.** *Let $f(\lambda) = \frac{A(\lambda)}{B(\lambda)}$, where $A(\lambda), B(\lambda)$ are analytic functions near 0 and $B(0) \neq 0$. If we know the first $k$-th coefficient of the Taylor series of $A(\lambda)$ and $B(\lambda)$, then we can calculate the first $k$-th coefficient of the Taylor series of $f(\lambda)$ in time $O(k^2)$.*

*Proof.* See Appendix B. □

**Theorem 5.4** (FPTAS)**.** *Fix $\beta > 1$, $\Delta \in N^+$, $\lambda \in \mathbb{D}$, there exists a FPTAS to compute the partition function of the Ising model $Z_G(\beta, \lambda)$.*

*Proof.* Denote $n = |V|$ and $m = |E|$. Let $E(G) = \{e_1, e_2, \cdots, e_m\}$ be the edge set of $G$ and define $E_i = \{e_1, e_2, \cdots, e_i\}$ for $(i = 1, 2, \cdots, m)$, and set $E_0 = \varnothing$. Note that

$$\frac{Z_G(\beta,\lambda)}{Z_{G\setminus E}(\beta,\lambda)} = \prod_{i=1}^{m} \frac{Z_{G\setminus\{e_1,e_2,\dots,e_{i-1}\}}(\beta,\lambda)}{Z_{G\setminus\{e_1,e_2,\dots,e_i\}}(\beta,\lambda)} = \frac{1}{\prod_{i=1}^{m} P_{G\setminus E_{i-1},e_i}(\beta,\lambda)}.$$

Since $Z_{G\setminus E} = (1+\lambda)^n$ (isolated vertices), to get a $(1+\varepsilon)$ approximation of $Z_G$, $(1 + \frac{\varepsilon}{(1+\varepsilon)m})$ approximation of $P_{G\setminus E_{i-1},e_i}(\beta, \lambda)$ is enough. By Lemma 5.1, the ratio has a constant lower bound $c$, thus $\frac{c\varepsilon}{(1+\varepsilon)m}$ additive error of the ratio is sufficient. By the uniform bound of the ratio in Lemma 4.5, the first $d$ terms of the Taylor series will give a $O(r^{-d})$ additive error for some $r > 1$ (the proof of Lemma 2.6 exactly shows this). Thus we need to calculate the first $O(\log(\frac{mC(1+\varepsilon)}{c\varepsilon})) = O(\log(n/\varepsilon))$ coefficients of the Taylor series of the ratio to get the $\frac{c\varepsilon}{(1+\varepsilon)m}$ additive error. By Lemma 5.2, it can be done in time $\text{poly}(n/\varepsilon)$. Then the FPTAS is established. □

**Remark 5.5.** *Though the FPTAS is established, we wonder can we get a FPTAS directly from the new SSM results like Weitz's algorithm?*

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## A Vertex-type SSM

### A.1 Vertex-type LDC

**Lemma A.1.** *For $\beta \geq 1$, $c \in [0,1)$, $G=(V,E)$ be a graph, $v \in V$, $A \subset V\backslash\{v\}$, $\boldsymbol{\lambda} \in \mathbb{D}^V$ with $\lambda \mid \lambda_v$ for all $v \in V$, $m = \{\lambda_v \to c\lambda_v\}$, $m_1 = \{\lambda_u \to c\lambda_u\}_{u\in A}$, then*

$$\lambda^{d_G(v,A)+1} \mid P_{G,m}(\beta,\boldsymbol{\lambda}) - P_{G,m}^{m_1}(\beta,\boldsymbol{\lambda}).$$

*Proof.*

$$\begin{aligned} P_{G,m}(\beta,\boldsymbol{\lambda}) - P_{G,m}^{m_1}(\beta,\boldsymbol{\lambda}) =& \frac{Z_G^m(\beta,\boldsymbol{\lambda})}{Z_G(\beta,\boldsymbol{\lambda})} - \frac{Z_G^{m,m_1}(\beta,\boldsymbol{\lambda})}{Z_G^{m_1}(\beta,\boldsymbol{\lambda})} \\ =& \frac{Z_G^m(\beta,\boldsymbol{\lambda})Z_G^{m_1}(\beta,\boldsymbol{\lambda}) - Z_G^{m,m_1}(\beta,\boldsymbol{\lambda})Z_G(\beta,\boldsymbol{\lambda})}{Z_G(\beta,\boldsymbol{\lambda})Z_G^{m_1}(\beta,\boldsymbol{\lambda})}. \end{aligned}$$

Clearly $\frac{1}{Z_G(\beta,\boldsymbol{\lambda})Z_G^{m_1}(\beta,\boldsymbol{\lambda})}$ is analytic near $\lambda = 0$. Combining with Lemma 3.1, we have $\lambda^{d_G(v,A)+1} \mid P_{G,m}(\beta,\boldsymbol{\lambda}) - P_{G,m}^{m_1}(\beta,\boldsymbol{\lambda})$. □

**Lemma A.2.** *For $\beta > 1$, $c \in [0,1)$, $G=(V,E)$ be a graph, $v \in V$, $A, B \subset V\backslash\{v\}$, $m = \{\lambda_v \to c\lambda_v\}$, $m_1 = \{\lambda_u \to c\lambda_u\}_{u\in A}$, $m_2 = \{\lambda_u \to c\lambda_u\}_{u\in B}$, then*

$$\lambda^{d_G(v,m_1\neq m_2)+1} \mid P_{G,m}^{m_1}(\beta,\lambda) - P_{G,m}^{m_2}(\beta,\lambda)$$

*where $m_1 \neq m_2$ is vertex set where $m_1$ and $m_2$ differ.*

*Proof.* Consider $\boldsymbol{\lambda}'$ as the uniform external field $\lambda^V$ changed by $m_1 \cap m_2$, $m_1' = m_1\backslash m_2$, $m_2' = m_2\backslash m_1$, then

$$\begin{aligned} P_{G,m}^{m_1}(\beta,\lambda) - P_{G,m}^{m_2}(\beta,\lambda) =& P_{G,m}^{m_1'}(\beta,\boldsymbol{\lambda}') - P_{G,m}^{m_2'}(\beta,\boldsymbol{\lambda}') \\ =& [P_{G,m}^{m_1'}(\beta,\boldsymbol{\lambda}') - P_{G,m}(\beta,\boldsymbol{\lambda}')] + [P_{G,m}(\beta,\boldsymbol{\lambda}') - P_{G,m}^{m_2'}(\beta,\boldsymbol{\lambda}')]. \end{aligned}$$

By the previous lemma, we have $\lambda^{d_G(v,m_1')+1} \mid P_{G,m_1'}(\beta,\boldsymbol{\lambda}') - P_{G,m}(\beta,\boldsymbol{\lambda}')$ and $\lambda^{d_G(v,m_2')+1} \mid P_{G,m}(\beta,\boldsymbol{\lambda}') - P_{G,m_2'}(\beta,\boldsymbol{\lambda}')$. Since $d_G(v, m_1 \neq m_2) = \min\{d_G(v,m_1'), d_G(v,m_2')\}$, we are done. □

### A.2 Uniform bound of vertex type ratio

**Lemma A.3** (c.f. Corollary 38 in [SY24])**.** *Let $G$ be a graph and $v$ be a vertex in $G$. Then the partition function of Ising model $Z_{G,v}^+(\beta,\boldsymbol{\lambda})$ can be expressed as:*

$$Z_{G,v}^+(\beta,\boldsymbol{\lambda}) = \lambda_v Z_{G\backslash\{v\}}(\beta,\boldsymbol{\lambda}^{v^+})$$

*where $Z_{G\backslash\{v\}}(\beta,\boldsymbol{\lambda}^{v^+})$ is the partition function of the Ising model with non-uniform external fields $\boldsymbol{\lambda}^{v^+}$ on the graph $G\backslash\{v\}$ obtained from $G$ by deleting $v$, and $\lambda_w^{v^+} = \lambda_w$ for $w \in V\backslash(N(v)\cup\{v\})$ and $\lambda_w^{v^+} = \beta\lambda_w$ for $w \in N(v)$.*

**Lemma A.4.** *Let $G=(V,E)$ be a graph, $\beta > 1$ , $\boldsymbol{\lambda} \in \mathbb{D}_{\frac{1}{\beta}}^V$, $v \in V(G)$, if $\lambda' \in \mathbb{D}_{\frac{1}{\beta}}$ and $\lambda_v' \neq \lambda_v$, then $P_{G,\{\lambda_v\to\lambda'\}}(\beta,\boldsymbol{\lambda})$ avoid $0$ and $1$.*

*Proof.* By Lee–Yang theorem, it is trivial that $P_{G,\{\lambda_v\to\lambda'\}}(\beta,\boldsymbol{\lambda}) \neq 0$. We prove the ratio avoid 1.

$$\begin{aligned}
&Z_G(\beta,\boldsymbol{\lambda}) - Z_G(\beta,\boldsymbol{\lambda}')\\
=&Z^+_{G,v}(\beta,\boldsymbol{\lambda}) + Z^-_{G,v}(\beta,\boldsymbol{\lambda}) - Z^+_{G,v}(\beta,\boldsymbol{\lambda}') - Z^-_{G,v}(\beta,\boldsymbol{\lambda}')\\
=&Z^+_{G,v}(\beta,\boldsymbol{\lambda}) - Z^+_{G,v}(\beta,\boldsymbol{\lambda}')\\
=&(\lambda_v - \lambda'_v)Z_{G\backslash\{v\}}(\beta,\boldsymbol{\lambda}^{v^+}) \quad \text{(Lemma A.3)}
\end{aligned}$$

Since $\boldsymbol{\lambda} \in \mathbb{D}^V_{\frac{1}{\beta}}$, then $\boldsymbol{\lambda}^{v^+} \in \mathbb{D}^{V\backslash\{v\}}$ , by Lee–Yang theorem, $Z_{G\backslash\{v\}}(\beta,\boldsymbol{\lambda}^{v^+}) \neq 0$, thus the ratio avoid 1.

□

**Lemma A.5.** *Fix $\beta \geq 1$ and $c \in [0,1)$, then for any compact set $S \subset \mathbb{D}_{\frac{1}{\beta}}\backslash\{0\}$, there exists a constant $C$ such that for any graph $G = (V,E)$, vertex $v \in V$, $A \subset V\backslash\{v\}$, $m = \{\lambda_v \to c\lambda_v\}$, $m_1 = \{\lambda_u \to c\lambda_u\}_{u\in A}$, such that $|P^{m_1}_{G,m}(\beta,\lambda)| \leq C$ for all $\lambda \in S$.*

*Proof.* By Lemma A.4, $P^{m_1}_{G,m}(\beta,\lambda)$ avoid 0 and 1 for all $\lambda \in \mathbb{D}_{\frac{1}{\beta}}\backslash\{0\}$. Pick a positive constant $\lambda' \in (0,\frac{1}{\beta})$, then $0 < P^{m_1}_{G,m}(\beta,\lambda') < 1$ always holds. Then by Lemma 2.7, the upper bound is got. □

Combining Lemmas A.2 and A.5 and the framework from LDC and uniform bound to SSM, we can establish the vertex type SSM.

**Theorem A.6** (vertex-type SSM)**.** *Fix edge activity $\beta \geq 1$ and uniform external $\lambda \in \mathbb{D}_{\frac{1}{\beta}}$ for Ising model, and $c \in [0,1)$. Then there exist constant $C > 0$ and $r > 1$ such that for all graph $G = (V,E)$, $v \in V$, $A,B \subset V\backslash\{v\}$, let $m = \{\lambda_v \to c\lambda\}$, $m_1 = \{\lambda_u \to c\lambda\}_{u\in A}$, $m_2 = \{\lambda_u \to c\lambda\}_{u\in B}$, we have*

$$\left|P^{m_1}_{G,m} - P^{m_2}_{G,m}\right| \leq Cr^{-d_G(v,m_1\neq m_2)}.$$

# B Proof of Lemma 5.2

*Proof.* Let $A(z) = \sum_{i=0}^{\infty} a_i z^i, B(z) = \sum_{i=0}^{\infty} b_i z^i$, $f(z) = \sum_{i=0}^{\infty} c_i z^i$ then

$$A(z) = \sum_{i=0}^{\infty} a_i z^i = B(z)f(z) = \sum_{i=0}^{\infty} b_i z^i \sum_{i=0}^{\infty} c_i z^i = \sum_{i=0}^{\infty}\sum_{j=0}^{i} b_j c_{i-j} z^i.$$

Thus we have $a_i = \sum_{j=0}^{i} b_j c_{i-j}$. Note $b_0 = B(0) \neq 0$, then $c_i = \frac{1}{b_0}(a_i - \sum_{j=1}^{i} b_j c_{i-j})$. This can be computed in time $O(k^2)$. □