Zero-Freeness and Deterministic Counting on Erdős–Rényi Random Graphs

Ke Shi
School of Computer Science and Technology
University of Science and Technology of China
self.ke.shi@gmail.com

Abstract

We establish complex zero-freeness and deterministic fully polynomial-time approximation schemes (FPTASes) for spin systems on the Erdős–Rényi random graph $G(n, d/n)$. For the antiferromagnetic q-state Potts model ($q \geq 3$) with $d < d_2(q) \sim 2q \log q$, the partition function is with high probability zero-free on a fixed-width neighborhood of $[0, 1]$, including proper q-colorings at $z = 0$; this yields a deterministic FPTAS for $q \geq (1/2 + \eta)d/\log d$ (fixed $\eta > 0$, large d), and, combined with the Achlioptas–Naor noncolorability bound, a deterministic FPTAS for counting proper q-colorings for all but at most one integer q at each fixed $d > 0$. An appendix extends the zero-freeness and the FPTAS up to the Erdős–Rényi condensation threshold of the Potts model, including proper colorings. For the zero-field Ising antiferromagnet and the zero-field Viana–Bray diluted spin glass on $G(n, d/n)$, we likewise obtain fixed-width zero-free neighborhoods and deterministic FPTASes up to their Kesten–Stigum thresholds. Dividing out each graph's forest weight removes the order-$\sqrt{n}$ fluctuations of the log-partition function caused by the random number of edges; complex moment bounds and Poisson short-cycle Taylor truncations then transfer nonvanishing from an explicit random cycle model, compensated where necessary, while connected-set Möbius inversion and Cayley's tree bound handle unbounded degrees in Barvinok interpolation.

Disclaimer: This work was developed through interaction with GPT-6 Astra, with Claude Opus 5.5 and Gemini 4 Argon used for writing and editing. While preparing this manuscript, we learned that Chen, Wang, and Zhang [CWZ26] had independently proved zero-freeness and deterministic counting for the hard-core and antiferromagnetic Potts models on random regular graphs via a closely similar argument. The results of our main text and Appendices A–B, treating $G(n, d/n)$ and the Viana–Bray spin glass, were obtained independently of [CWZ26], whereas Appendix C was added afterward, adapting their overlap-concentration strategy to reach Potts condensation on $G(n, d/n)$. A Lean 4 formalization of the zero-freeness results is available at https://github.com/KeShih/ER-zero-freeness.

1 Introduction

The partition function of a graphical spin system—such as proper q-colorings, the q-state Potts model, or the Ising model—is a polynomial or exponential sum in the interaction parameter that aggregates configuration weights on a graph G. In the classical program of Lee and Yang [YL52, LY52] and of Fisher [Fis65], phase transitions are governed by the zeros of the partition function in the complex plane: when a complex neighborhood of the physical parameter interval is free of zeros, the log-partition function is analytic and exhibits no phase transition.

On worst-case graphs of maximum degree Δ, complex zeros can approach the physical interval as soon as long-range correlations appear on the infinite Δ-regular tree, a barrier

known as the *tree-uniqueness threshold* (for instance, at $q \approx \Delta$ colors for proper colorings). On the Erdős–Rényi random graph $G(n, d/n)$ of fixed average degree d, however, typical instances behave very differently from worst-case graphs. Although $G(n, d/n)$ is locally tree-like, its random long-range connections rule out the adversarial boundary configurations that destroy correlation decay on worst-case trees. For example, $G(n, d/n)$ remains q-colorable with high probability down to $q \approx d/(2 \log d)$ [AN05], a factor of order $\log d$ below the tree-uniqueness scale $q \approx d$. Below condensation, real-valued partition functions normalized by their forest weight fluctuate at constant scale according to short cycles: for the Potts antiferromagnet at positive temperature on $G(n, d/n)$ below its condensation threshold [COEJ+18, Theorem 1.3], for q-colorings of $G(n, m)$ [Ras19, Theorem 1.1] (for large q up to the coloring condensation threshold, which agrees with the colorability threshold to first order), and for the Viana–Bray model throughout its annealed region $d \tanh^2 \beta < 1$ [GT04, Theorem 4].

Working in the complex plane on $G(n, d/n)$ poses two obstacles absent from bounded-degree or regular graph ensembles. First, because the number of edges in $G(n, d/n)$ fluctuates on scale $\sqrt{n}$, the unnormalized log-partition function fluctuates on scale $\sqrt{n}$ as well, so normalizing by a deterministic constant cannot isolate $O(1)$ cycle fluctuations. Second, the maximum degree of $G(n, d/n)$ grows unboundedly with n, so black-box bounded-degree coefficient algorithms [LSS19, PR17, PR19, YYZ22] cannot be applied directly in Barvinok's interpolation method [Bar16].

In this paper we show that complex zero-freeness and deterministic approximate counting on $G(n, d/n)$ persist far beyond the tree-uniqueness barrier across the second-moment regime. Dividing out the exact forest contribution of each graph eliminates edge-count fluctuations identically and leaves a centered polynomial normalized to 1 at the non-interacting point. Combining a complex moment bound (which controls the centered polynomial in probability on a complex domain) with the asymptotically Poisson counts of short cycles (which identify every fixed Taylor truncation at the non-interacting point with an explicit, almost surely nowhere-zero cycle model, compensated where necessary) transfers nonvanishing to the finite-graph partition function via Montel's compactness theorem, the identity theorem, and continuity of the minimum modulus. For counting, combining subset Möbius inversion over connected vertex sets with a deterministic layer-size cutoff replaces maximum-degree dependence by Cayley's tree bound on $G(n, d/n)$. Throughout, the graph (and any quenched edge signs) is random while the counting algorithm is completely deterministic: all model parameters and the average degree d are fixed before $n \to \infty$, and a single high-probability disorder event serves all allowed target parameters and relative precisions $\varepsilon \in (0, 1)$ simultaneously in time $O((n/\varepsilon)^C)$.

1.1 Potts zero-freeness and counting

For a finite graph $G = (V, E)$ and an integer $q \geq 2$, the q-state Potts partition function is

$$Z_G(q, z) = \sum_{\sigma : V \to [q]} z^{m_G(\sigma)}, \qquad m_G(\sigma) = \big|\{uv \in E : \sigma_u = \sigma_v\}\big|. \tag{1.1}$$

The regime $z \in [0, 1]$ is antiferromagnetic: $Z_G(q, 1) = q^{|V|}$ is the trivial high-temperature point, and $Z_G(q, 0)$ counts proper q-colorings of G (with the convention $0^0 = 1$). Let $G_n \sim G(n, d/n)$ be the Erdős–Rényi graph with fixed average degree $d > 0$. Define the second-moment density threshold $d_2(q)$ and the closed complex ρ-neighborhood K_ρ of $[0, 1]$ by

$$d_2(q) = \frac{2(q-1)^3 \log(q-1)}{q(q-2)}, \qquad K_\rho = \{z \in \mathbb{C} : \operatorname{dist}(z, [0, 1]) \leq \rho\}.$$

Theorem 1.1 (Potts zero-freeness). *Fix an integer $q \geq 3$ and $0 < d < d_2(q)$. There exists $\rho = \rho(q,d) > 0$, independent of n, such that*

$$\Pr[Z_{G_n}(q,z) \neq 0 \text{ for every } z \in K_\rho] \longrightarrow 1\,.$$

Theorem 1.2 (Deterministic Potts counting). *Fix an integer $q \geq 3$ and $0 < d < d_2(q)$. There is a deterministic algorithm which, given an n-vertex graph G, $x \in [0,1]$, and $\varepsilon \in (0,1)$, runs in $O((n/\varepsilon)^C)$ time for a constant $C = C(q,d)$ and outputs a number $\widetilde{Z}$ satisfying, with probability tending to one over $G_n \sim G(n,d/n)$,*

$$(1-\varepsilon)Z_{G_n}(q,x) \leq \widetilde{Z} \leq (1+\varepsilon)Z_{G_n}(q,x)$$

simultaneously for all $x \in [0,1]$ and $\varepsilon \in (0,1)$.

Here and below, time counts arithmetic operations; when the target parameter and ε are rational, the number of bit operations is at most this bound times a polynomial in their bit length (see the proof of Proposition 8.4). For a given quantity, a deterministic algorithm that runs in time $O((n/\varepsilon)^C)$ on every input, for a constant C, and whose output is, with probability tending to one, within a factor $1 \pm \varepsilon$ of the quantity simultaneously for all $\varepsilon \in (0,1)$, is called a deterministic fully polynomial-time approximation scheme (FPTAS) with high probability; Theorem 1.2 gives one for each $x \in [0,1]$, uniformly in x.

Because $d_2(q) \sim 2q\log q$ as $q \to \infty$, Theorems 1.1 and 1.2 reach the first-order colorability threshold $q \approx d/(2\log d)$ of $G(n,d/n)$, a factor of order $\log d$ below the tree-uniqueness threshold $q \approx d$.

Corollary 1.3 (Colorability-scale counting). *For every fixed $\eta > 0$ and all sufficiently large fixed d, Theorems 1.1 and 1.2 apply to every integer q satisfying*

$$q \geq \left(\frac{1}{2} + \eta\right)\frac{d}{\log d}\,.$$

The leading constant $1/2$ in Corollary 1.3 is sharp for zero-freeness on $[0,1]$: for $q \leq (1/2-\eta)d/\log d$ and large fixed d, G_n is non-q-colorable with high probability [AN05], so $Z_{G_n}(q,0) = 0$ and no zero-free neighborhood can contain $z = 0$. At $z = 0$, combining Theorem 1.2 below $d_2(q)$ with the constant-zero algorithm above the sufficient noncolorability threshold $(2q-1)\log q$ of Achlioptas and Naor [AN05] leaves at most one integer number of colors uncovered at any fixed average degree d.

Corollary 1.4 (At most one exceptional color number). *For every fixed $d > 0$ there exists an integer $q_d \geq 1$ such that, for every fixed integer $q \geq 1$ with $q \neq q_d$, the number of proper q-colorings of $G_n \sim G(n,d/n)$ admits a deterministic FPTAS with high probability.*

Corollaries 1.3 and 1.4 are proved in Appendix A. Restricting to a positive-temperature subinterval $[x_0,1]$ with $x_0 \in (0,1)$ (for $q \geq 3$) relaxes the density condition $d < d_2(q)$ to

$$d < d_2(q)\left(\frac{q-1+x_0}{(q-1)(1-x_0)}\right)^2, \tag{1.2}$$

yielding a fixed-width zero-free neighborhood of $[x_0,1]$ and a deterministic FPTAS on $[x_0,1]$; see Appendix B.

1.2 Ising antiferromagnet and Viana–Bray spin glass

For two-state spin systems on an n-vertex graph G equipped with edge signs $J = (J_e)_{e\in E(G)} \in \{-1,1\}^{E(G)}$, labeling spins by $\{-1,1\}$ and averaging over uniform spin configurations yields the centered edge-product polynomial

$$\mathcal{I}_{G,J}(t) = 2^{-n} \sum_{\sigma\in\{-1,1\}^{V(G)}} \prod_{uv\in E(G)} (1 + tJ_{uv}\sigma_u\sigma_v)\,. \tag{1.3}$$

Setting $J_e = 1$ for every edge recovers the zero-field Ising antiferromagnet ($q = 2$ in (1.1)) through the forest relation

$$Z_G(2,z) = 2^n \left(\frac{1+z}{2}\right)^{|E(G)|} \mathcal{I}_{G,1}\left(\frac{z-1}{z+1}\right)\,. \tag{1.4}$$

When $G_n \sim G(n,d/n)$ and its edges receive independent uniform signs $J_e \in \{-1,1\}$, (1.3) also governs the zero-field Viana–Bray diluted spin glass [GT04, VB85] at inverse temperature β: since $\exp(\beta s) = (\cosh\beta)(1 + s\tanh\beta)$ for $s \in \{-1,1\}$, its partition function satisfies

$$Z^{\mathrm{VB}}_{G,J}(\beta) = \sum_{\sigma\in\{-1,1\}^{V(G)}} \exp\left(\beta \sum_{uv\in E(G)} J_{uv}\sigma_u\sigma_v\right) = 2^n(\cosh\beta)^{|E(G)|}\,\mathcal{I}_{G,J}(\tanh\beta) \tag{1.5}$$

for all $\beta \in \mathbb{C}$ with $\cosh\beta \neq 0$.

Theorem 1.5 (Ising and Viana–Bray zero-freeness and counting)**.** *Fix $d > 0$, let $G_n \sim G(n,d/n)$, and let $J \in \{-1,1\}^{E(G_n)}$ assign independent uniform edge signs.*

(i) (Zero-field Ising antiferromagnet.) *For every $x_* \in (0,1)$ with $d((1-x_*)/(1+x_*))^2 < 1$, there exists $\rho = \rho(d,x_*) > 0$ such that, with probability tending to one, $Z_{G_n}(2,z) \neq 0$ whenever $\mathrm{dist}(z,[x_*,1]) \le \rho$.*

(ii) (Viana–Bray diluted spin glass.) *For every $\beta_* > 0$ with $d\tanh^2\beta_* < 1$, there exists $\rho = \rho(d,\beta_*) > 0$ such that, with probability tending to one, $\mathcal{I}_{G_n,J}(t) \neq 0$ whenever $|t| \le \tanh\beta_*$ and $Z^{\mathrm{VB}}_{G_n,J}(\beta) \neq 0$ whenever $\mathrm{dist}(\beta,[-\beta_*,\beta_*]) \le \rho$.*

Moreover, there exist deterministic algorithms which, given an n-vertex graph G (in (ii) *together with edge signs J), a target parameter in the stated interval, and $\varepsilon \in (0,1)$, run in time $O((n/\varepsilon)^C)$ and output a number $\widetilde{F}$ satisfying, with probability tending to one,*

$$(1-\varepsilon)F \le \widetilde{F} \le (1+\varepsilon)F$$

simultaneously across all $\varepsilon \in (0,1)$ and the stated interval: for $F = Z_{G_n}(2,x)$ on $x \in [x_*,1]$ with $C = C(d,x_*)$ in (i)*, and for $F = Z^{\mathrm{VB}}_{G_n,J}(\beta)$ on $\beta \in [-\beta_*,\beta_*]$ with $C = C(d,\beta_*)$ in* (ii)*.*

In part (i), the condition $d((1-x_*)/(1+x_*))^2 < 1$ holds for every $x_* \in (0,1)$ when $0 < d \le 1$, and reads $x_* > (\sqrt{d}-1)/(\sqrt{d}+1)$ when $d > 1$; this reaches the Kesten–Stigum and replica-symmetry threshold of the Ising antiferromagnet on $G(n,d/n)$ [COEJ+18], allowing $1 - x_*$ of order $d^{-1/2}$ rather than the uniqueness scale d^{-1}. In part (ii), $d\tanh^2\beta_* < 1$ covers the entire high-temperature replica-symmetric region of the zero-field Viana–Bray spin glass [GT04, VB85] (and all $\beta_* > 0$ when $0 < d \le 1$), giving a sparse diluted counterpart to the zero-freeness and deterministic counting results established up to the spin-glass transition for the dense Sherrington–Kirkpatrick model [BHL+26b, HLMM26, SZ26].

1.3 Proof structure and related work

Section 2 sets up the centered forest normalization for the Potts model, records its cycle factors, and collects the complex-analytic preliminaries (Cauchy estimates, Montel's theorem, and minimum-modulus continuity). Section 3 develops the general transfer theorem (Theorem 3.1) and constructs the compensated Poisson short-cycle limit $\mathcal{W}$ (Lemma 3.3), where analytic continuation of the deterministic cycle-mean series via $-\log(1-dt)$ removes the divergence of the uncompensated product beyond $d = q-1$ without altering any finite Taylor coefficient at $t = 0$. For the Potts model, Section 4 identifies the finite Taylor truncations via Poisson short-cycle counts; Section 5 proves a uniform complex first-moment bound via a balanced second-moment calculation and the sharp q-state relative-entropy inequality of Costeniuc, Ellis, and Touchette [CET05]; and Section 6 combines them into a single criterion (Proposition 6.1): a uniform first-moment bound on a complex tube around a subinterval $[-s, 0]$ of the antiferromagnetic t-interval with $s < 1$ and $ds^2 < 1$ yields zero-freeness with high probability on a thinner tube around it. This proves Theorem 1.1, and the same criterion serves the Ising antiferromagnet and Appendices B and C. Section 7 establishes the zero-freeness assertions of Theorem 1.5 for the Ising antiferromagnet and the Viana–Bray spin glass directly from untruncated complex second moments and Rademacher quadratic exponential bounds. Finally, Section 8 gives the connected-set Möbius coefficient algorithm and completes the Barvinok interpolation proofs of Theorems 1.2 and 1.5; while the short-cycle limit identifies fixed-order Taylor coefficients of the normalized polynomial in distribution for the zero-freeness proof, the algorithm computes the first $O(\log(n/\varepsilon))$ logarithmic Taylor coefficients of the same normalized polynomial at $t = 0$ exactly with high probability.

Previous deterministic approximate counting algorithms for q-colorings of $G(n, d/n)$ required $q \geq (1+\eta)d$ for free-energy approximation [Eft20], or $q \geq (2+\eta)d$ at large d for an FPTAS [LY26b]. For random sampling, rapid mixing of Glauber dynamics was recently proved down to $q \geq (1+o_d(1))d$ [LY26a]. Note that [LY26a, LY26b] accommodate broader interactions or pinned boundary conditions, whereas our results apply to the unpinned, zero-field partition functions on $G(n, d/n)$. On random regular bipartite graphs, FPTASes and samplers beyond tree uniqueness have been obtained via polymer expansions [CGŠV22, LLY26].

Real-valued distributional limits for normalized Potts, coloring, Ising, and diluted spin-glass partition functions are classic, via small-subgraph conditioning [COEJ+18, FL21, Ras19] (on $G(n, d/n)$, random regular graphs, and $G(n, m)$, respectively) or interpolation [GT04]; our cycle factors coincide with theirs. In particular, for $z = e^{-\beta} \in (0, 1)$ and d below the condensation threshold of [COEJ+18, equation (1.5)] (for $q \geq 3$ this is $d_{\mathrm{cond}}(q, z)$ in (C.1)), [COEJ+18, Theorem 1.3] uses the same per-graph normalization $q^n b(z)^{|E(G)|}$ (with b and t as in Section 2.1; the term $(n + \frac{1}{2}) \ln q$ printed in [COEJ+18, Theorem 1.3], which comes from [COEJ+18, equation (3.14)], should read $n \ln q$, as small d shows), and its limit law is that of $\log \mathcal{W}(t(z))$ from Lemma 3.3, compensating terms included; likewise, [GT04, Theorem 4] normalizes the Viana–Bray partition function by $2^n \prod_e \cosh(\beta J_e)$, and for $J_e = \pm 1$ and $d \tanh^2 \beta < 1$ its limit law is that of the logarithm of the signed cycle limit of Section 7 at $t = \tanh \beta$, up to the loop and double-edge terms of its multigraph. These are real-parameter results; here the limits are used as random analytic functions of complex t, identified through their Taylor coefficients at $t = 0$. In dense mean-field spin glasses, Bencs, Huang, Lee, Liu, and Regts [BHL+26b] combined complex second moments with Jensen's formula to obtain average-case zero-freeness and deterministic quasipolynomial-time approximation; subsequently, Harbuzova, Lepsveridze, Majid, and Moitra [HLMM26] achieved deterministic polynomial-time relative approximation for mixed p-spin models across the second-moment regime via color coding, and independently and concurrently Schmidhuber and Zlokapa [SZ26] obtained deterministic polynomial-time additive approximations to thermal expectations and log-partition functions up to the static

glass transition. Those works address fully connected mean-field ensembles, where the cycle statistics are Gaussian. On $G(n, d/n)$ they are discrete Poisson counts, which give a Poisson random analytic limit (exponentially compensated in the Potts and Ising cases); moreover, the edge count fluctuates on scale $\sqrt{n}$, and unbounded degrees require connected-set enumeration. Our analytic transfer framework builds on the convergence theory for random analytic functions [Shi12, Section 2] and on Coste's treatment of sparse random matrix characteristic polynomials via short-cycle limits, second-moment tightness, and exponential compensation [Cos23, Sections 5–6 and 8].

Concurrent work. Two concurrent works establish zero-freeness and deterministic approximate counting on uniform random Δ-regular graphs ($\Delta \geq 3$): Bencs, Huang, Lee, Liu, and Regts [BHL+26a, Theorems 1.1–1.2] for the zero-field antiferromagnetic Ising model strictly below the Kesten–Stigum threshold, and Chen, Wang, and Zhang [CWZ26] for the hard-core and antiferromagnetic Potts models (including proper q-colorings and the zero-field Ising antiferromagnet) throughout the interior of the replica-symmetric regime. The results in our main text and Appendices A–B were obtained independently of these works. On $G(n, d/n)$, we cover the Ising antiferromagnet and Viana–Bray spin glass strictly below their Kesten–Stigum thresholds, and $q \geq 3$ Potts for $d < d_2(q)$.

All three works use Barvinok interpolation with coefficients computed from small connected subgraphs. For zero-freeness, our approach shares with [CWZ26] the use of holomorphic tightness and a nonvanishing random short-cycle limit. Chen, Wang, and Zhang identify this limit through real-line small-subgraph conditioning; we use the Taylor coefficients at the non-interacting point. Bencs et al. use analytic short-cycle reweighting and Jensen's formula; their argument also shows that the expected number of zeros in the stated region tends to zero [BHL+26a, Section 3.3].

On $G(n, d/n)$, we use the graph-dependent forest factor to remove edge-count fluctuations, and Cayley's tree bound with deterministic enumeration cutoffs to handle unbounded degrees. For the two-state models, the normalized complex second moments remain bounded throughout the stated range without graph truncation. Appendix C was added after the appearance of [CWZ26] and adapts their overlap-concentration strategy to prove Theorem C.1: zero-freeness near $[x_*, 1]$ and a deterministic FPTAS on $[x_*, 1]$ for every $x_* \in [0, 1)$ and every d below the Erdős–Rényi Potts condensation threshold $d_{\mathrm{cond}}(q, x_*)$ of (C.1); since $d_2(q) \leq d_{\mathrm{cond}}(q, 0)$, the case $x_* = 0$ contains Theorems 1.1 and 1.2. These results address the antiferromagnetic Potts case of the Erdős–Rényi extension asked for in [CWZ26, Section 6].

AI disclosure. GPT-6 Astra was used in developing this work, and Claude Opus 5.5 and Gemini 4 Argon in writing and editing it. The author has verified all mathematical proofs and takes full responsibility for the content of this paper.

2 Preliminaries

We first factor out the leading tree contribution from the Potts partition function so that the remaining normalized polynomial depends only on cycles and subgraphs of higher excess. We then record the complex-analytic compactness and uniqueness facts used to pass from finite Taylor truncations to global zero-freeness.

2.1 Potts normalization and cycle factors

Throughout, $G = (V, E)$ denotes a finite simple undirected graph on $n = |V|$ vertices, and $[q] = \{1, \ldots, q\}$. In the Potts partition function (1.1), each edge contributes weight z

when its endpoints receive the same color and weight 1 otherwise; the interval $z \in [0,1]$ is antiferromagnetic, with $z = 0$ counting proper q-colorings and $z = 1$ giving q^n. In the Erdős–Rényi graph $G(n, d/n)$, edges are present independently with probability d/n, where $d > 0$ and q are fixed as $n \to \infty$; we consider $G(n, d/n)$ only for $n \geq d$, so suprema over n of quantities defined from G_n range over $n \geq d$.

To isolate the effect of cycles, we normalize $Z_G(q, z)$ by its value on a forest. Fix $q \geq 2$. Under independent uniform vertex colors, the endpoints of an edge agree with probability $1/q$, so the average weight of an edge is

$$b(z) = \frac{z}{q} + \frac{q-1}{q} = \frac{z+q-1}{q}\,.$$

Summing over the color of a leaf contributes $qb(z)$ for every color of its neighbor, and each isolated vertex contributes q. Repeated leaf removal therefore gives $Z_T(q, z) = q^{|V(T)|}b(z)^{|E(T)|}$ for every forest T. For a general graph G, we divide $Z_G(q, z)$ by $q^{|V|}b(z)^{|E|}$ to remove this tree factor.

Writing the normalized weight of an edge with endpoint colors $i, j \in [q]$ as one plus a mean-zero fluctuation gives

$$\frac{1 + (z-1)\mathbf{1}_{\{i=j\}}}{b(z)} = 1 + \frac{z-1}{z+q-1}(q\mathbf{1}_{\{i=j\}} - 1)\,.$$

For each fixed color i, the quantity $q\mathbf{1}_{\{i=j\}} - 1$ averages to zero over a uniform choice of $j \in [q]$. Its scalar prefactor defines the transformed interaction parameter $t = t(z) = (z-1)/(z+q-1)$. Let Id be the $q \times q$ identity matrix, let J be the $q \times q$ all-ones matrix, and put $A = q\,\mathrm{Id} - J$, so that the normalized edge weight is $1 + tA_{ij}$. Averaging over uniform colorings $\sigma \in [q]^{V(G)}$ yields the normalized partition polynomial

$$\widehat{Z}_G(t) = \mathbb{E}_\sigma \prod_{uv \in E(G)} (1 + tA_{\sigma_u\sigma_v})\,, \tag{2.1}$$

which is related to the original partition function by

$$\widehat{Z}_G(t(z)) = \frac{Z_G(q, z)}{q^{|V|}b(z)^{|E|}}\,. \tag{2.2}$$

The non-interacting point $z = 1$ maps to $t = 0$, where $\widehat{Z}_G(0) = 1$. Expanding the product over edges in (2.1) expresses $\widehat{Z}_G(t)$ as a generating function over edge subsets, with one power of t per selected edge. We therefore establish zero-freeness in the t-plane and pull the result back to the z-plane at the end.

Because independent components have independent colorings, $\widehat{Z}_G$ is multiplicative over connected components and satisfies $\widehat{Z}_T = 1$ on every forest T. The simplest subgraphs that make a nontrivial contribution are cycles. Labeling the vertices of an ℓ-cycle C_ℓ in cyclic order with $\sigma_{\ell+1} = \sigma_1$ and summing over all colorings gives

$$\widehat{Z}_{C_\ell}(t) = \frac{1}{q^\ell} \sum_{\sigma_1,\ldots,\sigma_\ell \in [q]} \prod_{i=1}^{\ell} (1 + tA_{\sigma_i\sigma_{i+1}}) = \operatorname{tr}\left[\left(\frac{J + tA}{q}\right)^\ell\right]\,.$$

Since J has eigenvalue q on the all-ones vector and 0 on its $(q-1)$-dimensional orthogonal complement, where A acts as $q\,\mathrm{Id}$, the matrix $(J + tA)/q$ has eigenvalue 1 with multiplicity 1 and eigenvalue t with multiplicity $q - 1$. Hence

$$\widehat{Z}_{C_\ell}(t) = 1 + (q-1)t^\ell\,. \tag{2.3}$$

As z ranges over the antiferromagnetic interval $[0,1]$, the parameter $t(z)$ ranges over

$$I = \left[-\frac{1}{q-1}, 0\right] .$$

For $r > 0$, we denote the open and closed r-neighborhoods of I in the complex plane by

$$U_r = \{t \in \mathbb{C} : \operatorname{dist}(t, I) < r\} , \qquad L_r = \overline{U_r} .$$

2.2 Analytic preliminaries

Short cycles in G_n determine each fixed Taylor truncation of $\widehat{Z}_{G_n}$ at $t = 0$. Passing from these finite Taylor truncations to nonvanishing on a complex neighborhood of I requires three classical facts from complex analysis [Con78, Chapters IV and VII]: Cauchy's coefficient estimates, Montel's compactness theorem, and the identity theorem.

We use the terms holomorphic and complex analytic interchangeably; locally uniform convergence means uniform convergence on every compact subset of the domain. For a function f holomorphic near t_0 and an integer $k \geq 1$, its degree-$(k-1)$ Taylor truncation at t_0 is

$$f^{[k]}(t) = \sum_{j=0}^{k-1} \frac{f^{(j)}(t_0)}{j!}(t - t_0)^j .$$

Lemma 2.1 (Cauchy estimates)**.** *Let $\Omega \subset \mathbb{C}$ be open, and let $\{w \in \mathbb{C} : |w - t_0| \leq r\} \subset \Omega$ with $r > 0$. For every holomorphic function f on Ω and every integer $j \geq 0$,*

$$\frac{f^{(j)}(t_0)}{j!} = \frac{1}{2\pi \mathrm{i}} \int_{|w-t_0|=r} \frac{f(w)}{(w - t_0)^{j+1}}\, dw , \qquad \frac{|f^{(j)}(t_0)|}{j!} \leq r^{-j} \max_{|w-t_0|=r} |f(w)| ,$$

where the circle is oriented counterclockwise. Moreover, every locally uniformly convergent sequence of holomorphic functions on Ω has a holomorphic limit, and its Taylor coefficients at t_0 converge to those of the limit.

Applied to the difference of two holomorphic functions, the Cauchy bound shows that uniform convergence on a surrounding circle controls each Taylor coefficient at t_0; see [Con78, Section IV.5 and Chapter VII].

Theorem 2.2 (Montel)**.** *Let $(f_n)_{n \geq 1}$ be a sequence of holomorphic functions on an open set $\Omega \subset \mathbb{C}$ such that, for every compact $K \subset \Omega$,*

$$\sup_n \sup_{t \in K} |f_n(t)| < \infty .$$

Then every subsequence of (f_n) has a further subsequence converging locally uniformly on Ω to a holomorphic function.

By Lemma 2.1 with $j = 1$, a uniform bound on a slightly larger compact set yields equicontinuity on K, so Theorem 2.2 follows from the Arzelà–Ascoli theorem and a diagonal extraction across a compact exhaustion of Ω [Con78, Chapter VII]. Crucially, the uniform bound must hold on an open complex neighborhood rather than on the real interval I alone.

Lemma 2.3 (Identity theorem)**.** *Let f and g be holomorphic on a connected open set $\Omega \subset \mathbb{C}$, and let $t_0 \in \Omega$. If $f^{(j)}(t_0) = g^{(j)}(t_0)$ for every integer $j \geq 0$, then $f = g$ throughout Ω.*

Equality of all derivatives at t_0 forces f and g to agree on the largest disk around t_0 in Ω, and analytic continuation along chains of overlapping disks propagates equality across the connected domain Ω [Con78, Chapter IV], even when the Taylor series at t_0 has radius of convergence smaller than the diameter of Ω.

Finally, on any nonempty compact set K, the minimum modulus is Lipschitz with respect to the supremum norm: for continuous functions f, g on K,

$$\left|\min_{t\in K}|f(t)| - \min_{t\in K}|g(t)|\right| \le \sup_{t\in K}|f(t)-g(t)|\,. \tag{2.4}$$

In particular, if f is continuous and nowhere zero on K, then $\min_{t\in K}|f(t)| > 0$, so every function uniformly close to f on K is also zero-free on K.

For random vectors or random functions, we write $X_n \xrightarrow{\mathrm{d}} X$ for convergence in distribution, meaning $\mathbb{E}h(X_n) \to \mathbb{E}h(X)$ for every bounded continuous real-valued function h. On spaces of Taylor truncations $f^{[k]}$ we use the Euclidean topology on the coefficient vector, and on spaces of holomorphic functions we use the topology of locally uniform convergence.

3 A transfer criterion and the cycle model

To prove that $\widehat{Z}_{G_n}$ has no zeros on a complex neighborhood of I with high probability, we combine two ingredients: a general transfer criterion that deduces zero-freeness from local boundedness in probability (implied by uniform first-moment bounds on compact sets) and convergence in law of finite Taylor truncations (Section 3.1), and an explicit nowhere-zero random analytic function whose Taylor truncations match the Poisson short-cycle limits (Section 3.2).

3.1 Analytic transfer from Taylor truncations

For an open set $\Omega \subseteq \mathbb{C}$, the space of holomorphic functions on Ω equipped with the locally uniform topology is complete and separable [Shi12, Section 2]. A random holomorphic function on Ω is a random element of this space with its Borel σ-field. A sequence of probability laws on this space is tight if for every $\epsilon > 0$ there is a compact subset carrying mass at least $1-\epsilon$ under every law in the sequence.

Theorem 3.1 (Zero-freeness from Taylor truncations)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, let $K \subset \Omega$ be a nonempty compact set, and fix an expansion point $t_0 \in \Omega$. Let $n_0 \ge 1$, let $(f_n)_{n\ge n_0}$ be a sequence of random holomorphic functions on Ω, and let f be a random holomorphic function on Ω. Assume that:*

(i) *For every compact $L \subset \Omega$, the sequence $\left(\sup_{t\in L}|f_n(t)|\right)_{n\ge n_0}$ is bounded in probability.*

(ii) *For every fixed integer $k \ge 1$, the Taylor truncations at t_0 satisfy $f_n^{[k]} \xrightarrow{\mathrm{d}} f^{[k]}$ as $n \to \infty$, in the joint distribution of their coefficient vectors.*

(iii) *With probability one, $f(t) \ne 0$ for every $t \in K$.*

Then $f_n \xrightarrow{\mathrm{d}} f$ in the locally uniform topology on Ω, and

$$\Pr[\text{there exists } t \in K \text{ with } f_n(t) = 0] \longrightarrow 0\,.$$

The following lemma gives a first-moment criterion for condition (i) and the tightness consequence needed in the proof.

Lemma 3.2 (First moments and tightness)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, let $n_0 \ge 1$, and let $(f_n)_{n\ge n_0}$ be a sequence of random holomorphic functions on Ω. If, for every compact $L \subset \Omega$,*

$$\sup_{n\ge n_0}\sup_{t\in L}\mathbb{E}|f_n(t)| < \infty\,,$$

then $\sup_{n\ge n_0}\mathbb{E}[\sup_{t\in L}|f_n(t)|]<\infty$ *for every compact* $L\subset\Omega$*; in particular, the sequence* $\left(\sup_{t\in L}|f_n(t)|\right)_{n\ge n_0}$ *is bounded in probability for every such* L*. Under local boundedness in probability on every compact subset of* Ω*, the laws of* (f_n) *are tight in the locally uniform topology.*

Proof. Since $g\mapsto\sup_{t\in L}|g(t)|$ is continuous for every compact $L\subset\Omega$ and $(g,w)\mapsto g(w)$ is jointly continuous on the product of the separable space of holomorphic functions on Ω with Ω, each $\sup_{t\in L}|f_n(t)|$ is a random variable and $(\omega,w)\mapsto|f_n(w)|$, with ω the sample point, is jointly measurable.

Suppose the first-moment bound holds on every compact set. Let $L\subset\Omega$ be compact, and choose $\delta>0$ so that the closed δ-neighborhood $L^\delta=\{w\in\mathbb{C}:\operatorname{dist}(w,L)\le\delta\}$ lies in Ω. For every $t\in L$, integrating the Cauchy integral formula of Lemma 2.1 ($j=0$) over radii $r\in(0,\delta)$ against $2r\,dr/\delta^2$ bounds $|f_n(t)|$ by the disk average $(\pi\delta^2)^{-1}\int_{|w-t|\le\delta}|f_n(w)|\,dA(w)\le(\pi\delta^2)^{-1}\int_{L^\delta}|f_n(w)|\,dA(w)$, where dA denotes planar Lebesgue measure [Shi12, Lemma 2.6]. Taking the supremum over $t\in L$ and applying Tonelli's theorem [Fol99, Theorem 2.37(a)] and the first-moment bound on L^δ gives

$$\sup_n\mathbb{E}\Big[\sup_{t\in L}|f_n(t)|\Big]\le\frac{1}{\pi\delta^2}\int_{L^\delta}\sup_n\mathbb{E}|f_n(w)|\,dA(w)<\infty\,, \tag{3.1}$$

so Markov's inequality gives the required boundedness in probability.

To prove tightness from local boundedness in probability, fix $\epsilon>0$ and let $(K_j)_{j\ge1}$ be a compact exhaustion of Ω. For each $j\ge1$ we may choose $M_j<\infty$ so that $\sup_n\Pr[\sup_{t\in K_j}|f_n(t)|>M_j]\le\epsilon2^{-j}$. The set of holomorphic functions on Ω satisfying $\sup_{t\in K_j}|g(t)|\le M_j$ for all $j\ge1$ has compact closure by Theorem 2.2, and by a union bound each f_n lies in this closure with probability at least $1-\epsilon$. Thus the laws of (f_n) are tight [Shi12, Proposition 2.5]. □

Proof of Theorem 3.1. Hypothesis (i) and Lemma 3.2 give tightness. By Prokhorov's theorem [Bil99, Theorem 5.1], every subsequence has a further subsequence converging in distribution to some random holomorphic function g on Ω. By Lemma 2.1, the map sending a holomorphic function to its jth Taylor coefficient at t_0 is continuous in the locally uniform topology, so hypothesis (ii) implies that the finite-dimensional distributions of the sequence of Taylor coefficients $(g^{(j)}(t_0)/j!)_{j\ge0}$ coincide with those of $(f^{(j)}(t_0)/j!)_{j\ge0}$. By Lemma 2.3, a holomorphic function on the connected domain Ω is uniquely determined by its Taylor coefficients at t_0; explicitly, successive power-series expansions along fixed chains of overlapping disks reconstruct point evaluations on a countable dense subset $S\subset\Omega$ as measurable functions of the coefficient sequence, so $(g(w))_{w\in S}$ and $(f(w))_{w\in S}$ have the same law. Every evaluation $h\mapsto h(w)$ with $w\in\Omega$ is a pointwise limit of evaluations at points of S, so g and f have the same finite-dimensional distributions and hence the same law [Shi12, Section 2]. The subsequence criterion [Bil99, Theorem 2.6] now yields $f_n\xrightarrow{\mathrm{d}}f$.

Finally, the functional $g\mapsto\min_{t\in K}|g(t)|$ is continuous by (2.4). The continuous mapping theorem and the Portmanteau theorem for the closed set $\{0\}$ [Bil99, Theorems 2.7 and 2.1] therefore give

$$\limsup_{n\to\infty}\Pr\Big[\min_{t\in K}|f_n(t)|=0\Big]\le\Pr\Big[\min_{t\in K}|f(t)|=0\Big]=0\,,$$

where the last equality holds by hypothesis (iii). □

3.2 The compensated short-cycle model

To apply Theorem 3.1, we must construct a nowhere-zero random holomorphic function f whose Taylor truncations at $t=0$ match the limiting Taylor truncations of $\widehat{Z}_{G_n}$. In Section 4

we show that, for $G_n \sim G(n, d/n)$, the numbers $C_\ell(G_n)$ of unoriented ℓ-cycles ($\ell \geq 3$) converge jointly in distribution to independent Poisson variables

$$P_\ell \sim \mathrm{Pois}(\lambda_\ell)\,, \qquad \lambda_\ell = \frac{d^\ell}{2\ell}\,,$$

and that the Taylor coefficients of $\widehat{Z}_{G_n}(t)$ at $t = 0$ through degree m converge jointly in law to those of the finite Poisson cycle product

$$Q_m(t) = \prod_{\ell=3}^{m} \bigl(1 + (q-1)t^\ell\bigr)^{P_\ell}\,,$$

where each ℓ-cycle contributes the factor $1 + (q-1)t^\ell$ from (2.3). Because factors with $\ell > m$ are $1 + O(t^{m+1})$, the Taylor truncation $Q_M^{[m+1]}$ at $t = 0$ is the same for all $M \geq m$. Our task is to pass to a limit as $m \to \infty$ on a complex neighborhood of $I = [-1/(q-1), 0]$ without altering these finite-degree Taylor coefficients.

Why the uncompensated product diverges. When $q - 1 < d < d_2(q)$, the uncompensated products $Q_m(t)$ fail to converge at the proper-coloring endpoint $t = -1/(q-1)$. To see where the divergence comes from, take the principal logarithm of $Q_m(t)$ near $t = 0$, expand $\log(1 + (q-1)t^\ell) = (q-1)t^\ell + O(t^{2\ell})$, and split each Poisson variable P_ℓ into its mean $\lambda_\ell = d^\ell/(2\ell)$ and its centered fluctuation $P_\ell - \lambda_\ell$:

$$\log Q_m(t) = \frac{q-1}{2} \sum_{\ell=3}^{m} \frac{(dt)^\ell}{\ell} + (q-1) \sum_{\ell=3}^{m} \left(P_\ell - \frac{d^\ell}{2\ell}\right) t^\ell + \sum_{\ell=3}^{m} P_\ell\, O(t^{2\ell})\,, \tag{3.2}$$

where on every compact subset of $\{t \in \mathbb{C} : |t| < 1,\ (q-1)|t|^3 < 1\}$ the implicit constant in $O(t^{2\ell})$ is independent of ℓ.

Under these modulus conditions, the centered and remainder series in (3.2) converge absolutely almost surely whenever $d|t|^2 < 1$, by the estimates leading to (3.5) below. At the coloring endpoint $t = -1/(q-1)$, the condition $d < d_2(q)$ ensures $d|t|^2 < 1$, because $\log x < (x - x^{-1})/2$ for $x > 1$ (the difference of the two sides vanishes at $x = 1$ and has derivative $(1 - x^{-1})^2/2$) and $(x - x^{-1})/2 = q(q-2)/(2(q-1))$ for $x = q - 1$ give

$$\frac{d_2(q)}{(q-1)^2} = \frac{2(q-1)\log(q-1)}{q(q-2)} < 1\,, \tag{3.3}$$

so both random sums converge almost surely. By contrast, the deterministic first sum in (3.2) converges absolutely only when $d|t| < 1$: for $q - 1 < d < d_2(q)$ and $t = -1/(q-1)$ we have $dt = -d/(q-1) < -1$, so its even and odd partial sums diverge to $+\infty$ and $-\infty$, and $Q_{2k}(t) \to \infty$ while $Q_{2k+1}(t) \to 0$.

Restoring the mean via analytic continuation. The decomposition (3.2) shows that only the deterministic mean series $\frac{q-1}{2}\sum_{\ell\geq 3}(dt)^\ell/\ell$ obstructs convergence in the regime $d|t| \geq 1$ with $d|t|^2 < 1$. Simply subtracting this series from $\log Q_m(t)$ would make the infinite product converge, but would change its Taylor coefficients at $t = 0$. Instead, we replace the finite deterministic sum in (3.2) by the analytic continuation of its infinite series. For $|dt| < 1$, the logarithmic series $-\log(1 - dt) = \sum_{\ell=1}^{\infty}(dt)^\ell/\ell$ gives

$$\frac{q-1}{2} \sum_{\ell=3}^{\infty} \frac{(dt)^\ell}{\ell} = -\frac{q-1}{2}\left[\log(1-dt) + dt + \frac{d^2t^2}{2}\right]\,,$$

where subtracting dt and $d^2t^2/2$ removes lengths $\ell = 1, 2$, which do not occur in a simple graph. The right-hand side extends holomorphically to the half-plane $\{\Re(1 - dt) > 0\}$, which contains $I = [-1/(q-1), 0]$ for every $d > 0$. We therefore define the compensated finite product by multiplying $Q_m(t)$ by the exponential of the tail of this series:

$$\mathcal{W}_m(t) = Q_m(t) \exp\left\{ -\frac{q-1}{2} \left[\log(1 - dt) + \sum_{\ell=1}^{m} \frac{(dt)^\ell}{\ell} \right] \right\}.$$

Since $\log(1 - dt) + \sum_{\ell=1}^{m} (dt)^\ell/\ell = O(t^{m+1})$ near $t = 0$, the compensating factor equals $1 + O(t^{m+1})$, so $\mathcal{W}_m$ and Q_m have identical Taylor coefficients through degree m. In $\log \mathcal{W}_m$, the deterministic partial sum in (3.2) is replaced by its analytic continuation above; the centered and remainder terms are unchanged.

Collecting the domain conditions needed for the principal logarithms and the convergence of the centered and quadratic series, we set

$$\mathcal{D} = \left\{ t \in \mathbb{C} : |t| < 1,\ d|t|^2 < 1,\ (q-1)|t|^3 < 1,\ \Re(1 - dt) > 0 \right\}.$$

Here $|t| < 1$ and $(q-1)|t|^3 < 1$ ensure that $|(q-1)t^\ell| < 1$ for every $\ell \geq 3$, so each cycle factor $1 + (q-1)t^\ell$ has positive real part, while $\Re(1 - dt) > 0$ ensures that $1 - dt$ has positive real part.

Lemma 3.3 (Compensated short-cycle model)**.** *Fix $d > 0$ and an integer $q \geq 2$, and define P_ℓ, Q_m, $\mathcal{W}_m$, and the domain $\mathcal{D}$ as above. Almost surely, the compensated products $\mathcal{W}_m$ converge locally uniformly on $\mathcal{D}$ to a holomorphic nowhere-zero function $\mathcal{W}$. Moreover, almost surely, for every fixed integer $m \geq 3$, the Taylor coefficients of $\mathcal{W}$ at $t = 0$ through degree m coincide with those of Q_m.*

Proof. Step 1: Locally uniform convergence and nonvanishing. By definition of $\mathcal{W}_m$ and Q_m, the logarithm of $\mathcal{W}_m$ on $\mathcal{D}$ obtained by summing the principal logarithms of $1 - dt$ and of the cycle factors is

$$\begin{aligned} \log \mathcal{W}_m(t) = &-\frac{q-1}{2} \left[\log(1 - dt) + dt + \frac{d^2t^2}{2} \right] \\ &+ \sum_{\ell=3}^{m} \left[P_\ell \log(1 + (q-1)t^\ell) - \frac{(q-1)(dt)^\ell}{2\ell} \right]. \end{aligned} \tag{3.4}$$

The first term on the right-hand side is deterministic and holomorphic on $\mathcal{D}$. To show that the sum in the second term converges locally uniformly on $\mathcal{D}$ almost surely, fix a nonempty compact subset $K \subset \mathcal{D}$ and put $T = \max_{t \in K} |t| < 1$ and $c = (q-1)T^3 < 1$. For $|a| \leq c < 1$, the Taylor series of $\log(1 + a)$ gives $|\log(1 + a) - a| \leq |a|^2/[2(1 - c)]$. Splitting each summand into $(q-1)(P_\ell - \lambda_\ell)t^\ell$ and $P_\ell[\log(1 + (q-1)t^\ell) - (q-1)t^\ell]$, using $\mathbb{E}P_\ell = \lambda_\ell$ and the Cauchy–Schwarz bound $\mathbb{E}|P_\ell - \lambda_\ell| \leq \sqrt{\lambda_\ell}$, and exchanging expectation and summation by the monotone convergence theorem for series [Fol99, Theorem 2.15] yields

$$\begin{aligned} &\mathbb{E}\left[\sum_{\ell=3}^{\infty} \sup_{t \in K} \left| P_\ell \log(1 + (q-1)t^\ell) - \frac{(q-1)(dt)^\ell}{2\ell} \right| \right] \\ &\qquad \leq \sum_{\ell=3}^{\infty} \left[\frac{(q-1)(\sqrt{d}\,T)^\ell}{\sqrt{2\ell}} + \frac{(q-1)^2(dT^2)^\ell}{4\ell(1-c)} \right] < \infty, \end{aligned} \tag{3.5}$$

where the series on the right converges because $dT^2 < 1$ on $K \subset \mathcal{D}$. Hence the series of supremum norms on K is finite almost surely, so the series in (3.4) converges absolutely and

uniformly on K. Exhausting $\mathcal{D}$ by a countable sequence of compact sets shows that, almost surely, $\log \mathcal{W}_m$ converges locally uniformly on $\mathcal{D}$ to a holomorphic limit, and therefore $\mathcal{W}_m$ converges locally uniformly on $\mathcal{D}$ to the holomorphic, nowhere-zero function $\mathcal{W}$ given by the exponential of that limit. On the remaining null event we set $\mathcal{W} := 1$, so that $\mathcal{W}$ is a random holomorphic function on $\mathcal{D}$.

Step 2: Preservation of Taylor coefficients. Fix $m \geq 3$. For every $M \geq m$, the compensating exponential factor in $\mathcal{W}_M$ equals $1 + O(t^{M+1})$ near $t = 0$, and each cycle factor in Q_M with $\ell > m$ equals $1 + O(t^{m+1})$. Consequently $\mathcal{W}_M(t) = Q_m(t) + O(t^{m+1})$ as $t \to 0$, so $\mathcal{W}_M$ and Q_m have the same Taylor coefficients at $t = 0$ through degree m. By Lemma 2.1, locally uniform convergence $\mathcal{W}_M \to \mathcal{W}$ as $M \to \infty$ preserves each Taylor coefficient at $t = 0$. □

4 Short cycles and Taylor coefficients

We first verify condition (ii) of Theorem 3.1 by showing that each fixed Taylor truncation of $\widehat{Z}_{G_n}$ at $t = 0$ is determined with high probability by the short cycles of G_n. Throughout this section, $d > 0$ and the integer $q \geq 2$ are arbitrary and fixed, $C_\ell(G)$ denotes the number of unoriented simple ℓ-cycles in G, and $(P_\ell)_{\ell \geq 3}$ are independent Poisson random variables with means $\lambda_\ell = d^\ell/(2\ell)$.

To prove that the short cycle counts converge jointly to independent Poisson variables, we use the method of factorial moments [JŁR00, Theorem 6.10]. For a real number x and an integer $k \geq 1$, define the falling factorial $(x)_k = x(x-1)\cdots(x-k+1)$, with $(x)_0 = 1$. Because a $\mathrm{Pois}(\lambda)$ random variable P satisfies $\mathbb{E}[(P)_k] = \lambda^k$ for every $k \geq 0$, independent Poisson variables $(P_3, \ldots, P_m)$ have joint factorial moments $\mathbb{E}\prod_{\ell=3}^m (P_\ell)_{k_\ell} = \prod_{\ell=3}^m \lambda_\ell^{k_\ell}$, and convergence of all joint factorial moments of $(C_3(G_n), \ldots, C_m(G_n))$ to these products implies joint convergence in distribution. Combinatorially, $(C_\ell(G_n))_{k_\ell}$ counts ordered k_ℓ-tuples of distinct ℓ-cycles in G_n: vertex-disjoint cycle tuples contribute the main term $\prod_{\ell=3}^m \lambda_\ell^{k_\ell}$, whereas tuples sharing a vertex span more edges than vertices and vanish in expectation as $O(n^{-1})$.

Lemma 4.1 (Short cycle counts)**.** *For every fixed integer $m \geq 3$, as $n \to \infty$,*

$$(C_3(G_n), \ldots, C_m(G_n)) \xrightarrow{\mathrm{d}} (P_3, \ldots, P_m)\,.$$

Proof. Fix nonnegative integers $k_3, \ldots, k_m$. By the factorial-moment criterion [JŁR00, Theorem 6.10], it suffices to show that

$$\mathbb{E}\prod_{\ell=3}^m \big(C_\ell(G_n)\big)_{k_\ell} = \prod_{\ell=3}^m \lambda_\ell^{k_\ell} + O(n^{-1})\,. \tag{4.1}$$

Put $L = \sum_{\ell=3}^m \ell k_\ell$. Expanding $\prod_{\ell=3}^m (C_\ell(G_n))_{k_\ell}$ counts tuples of distinct unoriented cycles in G_n, with k_ℓ ordered cycles of length ℓ for each $3 \leq \ell \leq m$. We partition all such candidate tuples in the complete graph K_n on $[n]$ into vertex-disjoint selections and intersecting selections.

In a vertex-disjoint selection, the L cycle vertices can be chosen in order in $(n)_L$ ways. Because each unoriented ℓ-cycle corresponds to 2ℓ cyclic vertex sequences (by choice of starting vertex and orientation) and the k_ℓ cycles of length ℓ are ordered, there are $(n)_L / \prod_{\ell=3}^m (2\ell)^{k_\ell}$ vertex-disjoint selections. Each requires L edges of G_n, which appear with probability $(d/n)^L$, so vertex-disjoint selections contribute

$$\frac{(n)_L}{\prod_{\ell=3}^m (2\ell)^{k_\ell}} \left(\frac{d}{n}\right)^L = \frac{(n)_L}{n^L} \prod_{\ell=3}^m \lambda_\ell^{k_\ell} = \prod_{\ell=3}^m \lambda_\ell^{k_\ell} + O(n^{-1})\,.$$

If at least two selected cycles share a vertex, let H be the union of all selected cycles. Every vertex of H has degree at least 2, and at least one connected component of H contains two distinct cycles and therefore is not a single cycle, so $|E(H)| > |V(H)|$. For each fixed isomorphism type of H with $v = |V(H)| \le L$ and $e = |E(H)| \ge v+1$, there are $O(n^v)$ labeled copies of H in K_n, each supporting $O(1)$ cycle tuples and appearing in G_n with probability $(d/n)^e$, giving an expected contribution of $O(n^v(d/n)^e) = O(n^{-1})$. Summing over the finitely many isomorphism types for fixed $m, k_3, \dots, k_m$ shows that intersecting selections contribute $O(n^{-1})$ in expectation, proving (4.1). □

For a subset of edges $S \subseteq E(G)$, let its support be the subgraph of G spanned by S and its incident vertices. A connected graph of minimum degree at least 2 has at least as many edges as vertices, with equality if and only if it is a simple cycle. Thus every connected support H of minimum degree at least 2 that is not a cycle has positive excess $|E(H)| - |V(H)| \ge 1$; hence, for each fixed isomorphism type, G_n contains a copy of H with probability $O(n^{-1})$.

Lemma 4.2 (Taylor coefficients from short cycles)**.** *Fix an integer $m \ge 3$. With probability $1 - O_m(n^{-1})$,*

$$\widehat{Z}_{G_n}(t) \equiv \prod_{\ell=3}^{m} (1 + (q-1)t^\ell)^{C_\ell(G_n)} \pmod{t^{m+1}}.$$

Consequently, as $n \to \infty$, the Taylor coefficients $(\widehat{Z}^{(j)}_{G_n}(0)/j!)_{0 \le j \le m}$ converge jointly in distribution to the corresponding Taylor coefficients at $t = 0$ of

$$\prod_{\ell=3}^{m} (1 + (q-1)t^\ell)^{P_\ell}.$$

Proof. Expanding the edge product in (2.1) over subsets $S \subseteq E(G)$ gives

$$\widehat{Z}_G(t) = \sum_{S \subseteq E(G)} w(S)\, t^{|S|}, \qquad w(S) = \mathbb{E}_\sigma \prod_{uv \in S} A_{\sigma_u \sigma_v}.$$

If the support of S has a vertex v of degree 1, with unique incident edge $uv \in S$, then averaging over σ_v annihilates $A_{\sigma_u\sigma_v}$ because each row of A sums to zero, so $w(S) = 0$. Moreover, $w(S)$ is multiplicative over connected components of the support of S, and for an ℓ-cycle its weight is $\operatorname{tr}(A^\ell)/q^\ell = q - 1$ by (2.3).

There are only finitely many isomorphism types of connected graphs H with $|V(H)| < |E(H)| \le 2m$, and each has expected copy count $O(n^{|V(H)|}(d/n)^{|E(H)|}) = O(n^{-1})$ in G_n. By Markov's inequality and a union bound, with probability $1 - O_m(n^{-1})$ the graph G_n contains no such subgraph. On this event, any two distinct cycles of length at most m are vertex-disjoint (otherwise their union would satisfy $|V(H)| < |E(H)| \le 2m$), and every edge set $S \subseteq E(G_n)$ with $|S| \le m$ and $w(S) \ne 0$ has minimum degree at least 2 and no component with more edges than vertices, so its support is a vertex-disjoint union of cycles of total length at most m. If S is the union of k such cycles, multiplicativity gives $w(S) = (q-1)^k$. Comparing coefficients of t^j for $0 \le j \le m$ proves the polynomial congruence modulo t^{m+1}.

Finally, for nonnegative integers $x_3, \dots, x_m$, each coefficient of $\prod_{\ell=3}^m (1 + (q-1)t^\ell)^{x_\ell}$ of degree at most m is a polynomial in $(x_3, \dots, x_m)$ via $\binom{x_\ell}{r} = (x_\ell)_r / r!$. By Lemma 4.1 and the continuous mapping theorem [Bil99, Theorem 2.7], the degree-m Taylor truncation of $\prod_{\ell=3}^m (1 + (q-1)t^\ell)^{C_\ell(G_n)}$ converges in distribution to that of $\prod_{\ell=3}^m (1 + (q-1)t^\ell)^{P_\ell}$, and the high-probability agreement modulo t^{m+1} transfers this limit to $\widehat{Z}^{[m+1]}_{G_n}$. □

5 Uniform complex moment bounds

We now prove a uniform first-moment bound for $\widehat{Z}_{G_n}$, which gives hypothesis (i) of Theorem 3.1 by Lemma 3.2. Throughout Section 5 and in the proof of Theorem 1.1, $q \geq 3$ and $0 < d < d_2(q)$ are fixed, and implicit constants may depend on them; Proposition 6.1 in Section 6 is stated for every $q \geq 2$ and $d > 0$. We bound $\mathbb{E}|\widehat{Z}_{G_n}(t)|$ uniformly on a closed complex tube $L_{4\rho}$ around $I = [-1/(q-1), 0]$. Bounding the second moment of $\widehat{Z}_{G_n}(t)$ over all colorings at once would fail because colorings with strongly unbalanced color classes distort the conditional relative-entropy bound in Section 5.2. We therefore proceed in two stages: in Section 5.1 we restrict the coloring sum to a balanced window $\mathcal{B}_n$ at an exponentially small L^1 cost, and in Section 5.2 we bound the complex second moment of the restricted polynomial $\widehat{Z}^{\mathrm{B}}_{G_n}(t)$ by comparing the two-replica edge-weight growth against the relative entropy of the overlap matrix.

5.1 Balancing the color sum

In the second-moment calculation of Section 5.2, applying the q-state relative-entropy inequality to the conditional distribution of the second coloring given the first multiplies the conditional entropy by the first coloring's class fractions $q\alpha_i(\sigma)$, where $\alpha_i(\sigma) = |\sigma^{-1}(i)|/n$ is the proportion of vertices assigned color i. To keep these factors close to 1, we fix a tolerance $\delta \in (0, 1/(2q))$ (chosen in Section 5.2) and retain only colorings in the balanced window

$$\mathcal{B}_n = \{\sigma \in [q]^{V(G)} : |\alpha_i(\sigma) - 1/q| \leq \delta \text{ for all } i \in [q]\}\,.$$

For an n-vertex graph G, the corresponding balanced partition polynomial is

$$\widehat{Z}^{\mathrm{B}}_G(t) = \mathbb{E}_\sigma\left[\mathbf{1}_{\{\sigma\in\mathcal{B}_n\}} \prod_{uv\in E(G)} (1 + tA_{\sigma_u\sigma_v})\right]\,.$$

Because the indicator $\mathbf{1}_{\{\sigma\in\mathcal{B}_n\}}$ does not depend on t, the function $\widehat{Z}^{\mathrm{B}}_G$ remains a polynomial in t.

Under the uniform distribution on $[q]^n$, a coloring falls outside $\mathcal{B}_n$ with probability $O(e^{-2\delta^2 n})$, whereas moving t to complex distance 4ρ from the real interval I increases each edge factor by at most $4(q-1)\rho$, so the expected product over edges of $G_n \sim G(n, d/n)$ is at most $e^{d/2+2d(q-1)\rho n}$. Thus the rare-event probability dominates the complex weight growth whenever

$$2d(q-1)\rho < \delta^2\,. \tag{5.1}$$

Here ρ is a t-plane parameter (the moment bound below holds on the tube $L_{4\rho}$ of half-width 4ρ), distinct from the z-plane width in Theorem 1.1. The first-moment bound (5.11) on $L_{4\rho}$ proved below is the hypothesis of Proposition 6.1 for the interval I, which deduces zero-freeness on a thinner tube.

Lemma 5.1 (Pointwise remainder)**.** *Under (5.1), there exists $C = C(q, d, \delta, \rho) < \infty$ such that, for all n,*

$$\sup_{t\in L_{4\rho}} \mathbb{E}|\widehat{Z}_{G_n}(t) - \widehat{Z}^{\mathrm{B}}_{G_n}(t)| \leq Ce^{-\delta^2 n}\,.$$

Proof. Taking absolute values inside the expectation over colorings and using independence of σ and G_n separates the probability of an unbalanced coloring from its expected graph weight: for every $t \in L_{4\rho}$,

$$\mathbb{E}|\widehat{Z}_{G_n}(t) - \widehat{Z}^{\mathrm{B}}_{G_n}(t)| \leq \Pr_\sigma[\sigma \notin \mathcal{B}_n] \max_{\sigma\in[q]^n} \mathbb{E}_{G_n} \prod_{uv\in E(G_n)} |1 + tA_{\sigma_u\sigma_v}|\,.$$

By Hoeffding's inequality [Hoe63, Theorem 2] applied to each of the q binomial color counts $n\alpha_i(\sigma) \sim \mathrm{Bin}(n, 1/q)$ and a union bound, $\Pr_\sigma[\sigma \notin \mathcal{B}_n] \le 2qe^{-2\delta^2 n}$.

To bound the expected graph weight of a fixed coloring σ, choose $x \in I = [-1/(q-1), 0]$ with $|t - x| \le 4\rho$. Since $1 + xA_{ij} \ge 0$ for $x \in I$ and $|A_{ij}| \le q - 1$, the triangle inequality gives

$$|1 + tA_{ij}| \le 1 + xA_{ij} + 4(q-1)\rho\,.$$

Summing $A_{\sigma_u\sigma_v} = q\mathbf{1}_{\{\sigma_u=\sigma_v\}} - 1$ over all unordered pairs $u < v$ and using Cauchy–Schwarz ($\sum_{i=1}^q \alpha_i(\sigma)^2 \ge 1/q$) yields

$$\sum_{u<v} A_{\sigma_u\sigma_v} = \frac{n^2}{2}\left(q\sum_{i=1}^q \alpha_i(\sigma)^2 - 1\right) - \frac{n(q-1)}{2} \ge -\frac{n(q-1)}{2}\,.$$

Because $-1/(q-1) \le x \le 0$, it follows that $x\sum_{u<v} A_{\sigma_u\sigma_v} \le n/2$. By independence of the edges of G_n and the inequality $1 + y \le e^y$, we obtain

$$\begin{aligned}
\mathbb{E}_{G_n} \prod_{uv\in E(G_n)} |1 + tA_{\sigma_u\sigma_v}| &= \prod_{u<v}\left[1 + \frac{d}{n}\big(|1 + tA_{\sigma_u\sigma_v}| - 1\big)\right] \\
&\le \exp\left\{\frac{d}{n}\sum_{u<v}\big(xA_{\sigma_u\sigma_v} + 4(q-1)\rho\big)\right\} \\
&\le \exp\left\{\frac{d}{2} + 2d(q-1)\rho(n-1)\right\}\,,
\end{aligned}$$

uniformly in $\sigma \in [q]^n$ and $t \in L_{4\rho}$. Combining this bound with the probability estimate $\Pr_\sigma[\sigma \notin \mathcal{B}_n] \le 2qe^{-2\delta^2 n}$ and using (5.1) gives

$$\sup_{t\in L_{4\rho}} \mathbb{E}\big|\widehat{Z}_{G_n}(t) - \widehat{Z}^{\mathrm{B}}_{G_n}(t)\big| \le 2qe^{d/2}e^{-[2\delta^2 - 2d(q-1)\rho]n} \le 2qe^{d/2}e^{-\delta^2 n}\,. \qquad \square$$

5.2 The second-moment bound

For a fixed graph G, multiplying the balanced color average $\widehat{Z}^{\mathrm{B}}_G(t)$ by its complex conjugate expresses $|\widehat{Z}^{\mathrm{B}}_G(t)|^2$ as an expectation over two independent uniform colorings $\sigma, \tau \in [q]^{V(G)}$. Since the matrix A has real entries,

$$|\widehat{Z}^{\mathrm{B}}_G(t)|^2 = \mathbb{E}_{\sigma,\tau}\left[\mathbf{1}_{\{\sigma,\tau\in\mathcal{B}_n\}} \prod_{uv\in E(G)} (1 + tA_{\sigma_u\sigma_v})(1 + \bar{t}A_{\tau_u\tau_v})\right]\,. \tag{5.2}$$

Averaging (5.2) over $G_n \sim G(n, d/n)$ (on vertex set $[n]$) couples the two colorings through their empirical overlap matrix $R = (R_{ij})_{i,j\in[q]}$, defined by

$$R_{ij} = \frac{1}{n}\big|\{v \in [n] : (\sigma_v, \tau_v) = (i, j)\}\big| \qquad (i, j \in [q])\,. \tag{5.3}$$

Because $\sum_{i,j=1}^q R_{ij} = 1$, the matrix R is a probability distribution on $[q]^2$—namely, the joint law of (σ_v, τ_v) at a uniformly chosen vertex $v \in [n]$—with row and column marginal distributions $\alpha = (\alpha_i)_{i\in[q]}$ and $\beta = (\beta_j)_{j\in[q]}$ given by

$$\alpha_i = \sum_{j=1}^q R_{ij} = \alpha_i(\sigma)\,, \qquad \beta_j = \sum_{i=1}^q R_{ij} = \alpha_j(\tau)\,.$$

Correlated pairs of colorings have large expected edge weight but small multinomial probability. Two external inputs quantify this competition: the sharp relative-entropy inequality for the Curie–Weiss–Potts model [CET05] and Agrawal's exponential-moment bound for empirical relative entropy [Agr20, Theorem I.3].

Lemma 5.2 (Entropy comparison)**.** *Let $u = (1/q, \dots, 1/q)$ be the uniform distribution on $[q]$. For every probability vector $p = (p_1, \dots, p_q)$,*

$$D(p\|u) := \sum_{i=1}^{q} p_i \log(qp_i) \geq c_q \left(q \sum_{i=1}^{q} p_i^2 - 1 \right), \qquad c_q = \frac{(q-1)\log(q-1)}{q(q-2)},$$

with $0 \log 0 = 0$.

Proof. By the variational characterization of the Curie–Weiss–Potts free energy at the critical inverse temperature $\beta_c = 2(q-1)\log(q-1)/(q-2) = 2qc_q$ [CET05, Theorem 3.1(b), equations (2.3) and (3.1)], the uniform distribution u minimizes $D(p\|u) - (\beta_c/2)\sum_{i=1}^q p_i^2 = D(p\|u) - qc_q \sum_{i=1}^q p_i^2$ over the simplex. Comparing the value of this functional at p with its value at u gives the inequality. □

Theorem 5.3 (Multinomial exponential moment [Agr20, Theorem I.3])**.** *Let $n \geq 1$ and $k \geq 2$ be integers, let $p = (p_1, \dots, p_k)$ be a probability vector with positive entries, and let $X \sim \mathrm{Mult}(n, p)$. For every $s \in [0, 1)$,*

$$\mathbb{E}[\exp\{snD(X/n\|p)\}] \leq \frac{1}{(1-s)^{k-1}}, \tag{5.4}$$

where $D(r\|p) = \sum_{i=1}^k r_i \log(r_i/p_i)$, with $0 \log 0 = 0$.

Lemma 5.4 (Uniform second moment)**.** *Assume $d < d_2(q)$. There exist $\delta \in (0, 1/(2q))$ and $\rho > 0$ such that*

$$\sup_n \sup_{t \in L_{4\rho}} \mathbb{E}\left|\widehat{Z}^{\mathrm{B}}_{G_n}(t)\right|^2 < \infty.$$

Moreover, δ and ρ satisfy (5.1).

Proof. Step 1: Two-replica edge-weight bound. Fix colorings $\sigma, \tau \in \mathcal{B}_n$ with overlap distribution R and marginals α, β as in (5.3). Averaging the product over edges in (5.2) with respect to $G_n \sim G(n, d/n)$ gives

$$\prod_{u<v} \left\{ 1 + \frac{d}{n}\left[(1 + tA_{\sigma_u\sigma_v})(1 + \bar{t}A_{\tau_u\tau_v}) - 1\right] \right\}.$$

To bound the modulus of each factor, apply $|1+\zeta| \leq \exp(\Re\zeta + |\zeta|^2/2)$. Because the bracketed term is bounded uniformly on $L_{4\rho}$, the factor d/n makes each increment $O(n^{-1})$, so summing the quadratic terms over all $\binom{n}{2}$ pairs contributes $\binom{n}{2} \cdot O(n^{-2}) = O(1)$ in the exponent.

For the linear term, let $\mathbb{E}_{u,v}$ denote expectation over independent uniform vertices $u, v \in [n]$ sampled with replacement, so that (σ_u, τ_u) and (σ_v, τ_v) are independent with law R. Replacing $\sum_{u<v}$ by $\frac{n^2}{2}\mathbb{E}_{u,v}$ adds only the n diagonal terms $u = v$, changing the exponent by $\frac{d}{n} \cdot O(n) = O(1)$. Since A has real entries,

$$\begin{aligned} &\Re\mathbb{E}_{u,v}\left[(1 + tA_{\sigma_u\sigma_v})(1 + \bar{t}A_{\tau_u\tau_v}) - 1\right] \\ &\quad = (\Re t)\,\mathbb{E}_{u,v}[A_{\sigma_u\sigma_v}] + (\Re t)\,\mathbb{E}_{u,v}[A_{\tau_u\tau_v}] + |t|^2\,\mathbb{E}_{u,v}[A_{\sigma_u\sigma_v}A_{\tau_u\tau_v}]. \end{aligned} \tag{5.5}$$

Because $A_{ab} = q\mathbf{1}_{\{a=b\}} - 1$, the two single-replica expectations in (5.5) are nonnegative:

$$\mathbb{E}_{u,v}[A_{\sigma_u\sigma_v}] = q\sum_{i=1}^{q} \alpha_i^2 - 1 = q\sum_{i=1}^{q} (\alpha_i - 1/q)^2 \geq 0,$$

and likewise $\mathbb{E}_{u,v}[A_{\tau_u\tau_v}] = q\sum_{j=1}^q(\beta_j - 1/q)^2 \geq 0$. Writing $A_{ab} = q\sum_{i=1}^q(\mathbf{1}_{\{a=i\}} - 1/q)(\mathbf{1}_{\{b=i\}} - 1/q)$ shows that the two-replica product expectation is also a sum of squares:

$$\mathbb{E}_{u,v}[A_{\sigma_u\sigma_v}A_{\tau_u\tau_v}] = q^2\sum_{i,j=1}^q\left(R_{ij} - \frac{\alpha_i}{q} - \frac{\beta_j}{q} + \frac{1}{q^2}\right)^2 \geq 0\,.$$

On $L_{4\rho}$ we have $|t| \leq T = 1/(q-1) + 4\rho$ and $\Re t \leq 4\rho$. Imposing $4\rho \leq T^2$ bounds all three coefficients $\Re t$, $\Re t$, and $|t|^2$ in (5.5) by T^2. Since the three expectations are nonnegative, we may replace each coefficient by T^2 and recombine the three terms using $1 + A_{ab} = q\mathbf{1}_{\{a=b\}}$:

$$\begin{aligned}\Re\mathbb{E}_{u,v}[(1 + tA_{\sigma_u\sigma_v})(1 + \bar{t}A_{\tau_u\tau_v}) - 1] &\leq T^2\,\mathbb{E}_{u,v}[A_{\sigma_u\sigma_v} + A_{\tau_u\tau_v} + A_{\sigma_u\sigma_v}A_{\tau_u\tau_v}]\\ &= T^2\,\mathbb{E}_{u,v}[(1 + A_{\sigma_u\sigma_v})(1 + A_{\tau_u\tau_v}) - 1]\\ &= T^2\left(q^2\sum_{i,j=1}^q R_{ij}^2 - 1\right).\end{aligned}$$

Multiplying by $dn/2$ and absorbing the $O(1)$ quadratic and diagonal errors into a prefactor $C < \infty$ independent of n and $t \in L_{4\rho}$ yields

$$\mathbb{E}|\widehat{Z}^{\mathrm{B}}_{G_n}(t)|^2 \leq C\mathbb{E}_{\sigma,\tau}\left[\mathbf{1}_{\{\sigma,\tau\in\mathcal{B}_n\}}\exp\left\{\frac{dnT^2}{2}\left(q^2\sum_{i,j=1}^q R_{ij}^2 - 1\right)\right\}\right]. \tag{5.6}$$

Step 2: Relative-entropy comparison and parameter choice. For $\sigma \in \mathcal{B}_n$, each color class has $\alpha_i \geq 1/q - \delta > 0$; let $p^{(i)}$ be the conditional distribution of τ on color class i of σ, defined by $p^{(i)}_j = R_{ij}/\alpha_i$, and let U be the uniform distribution on $[q]^2$. Since $\sigma \in \mathcal{B}_n$ implies $q\alpha_i \leq 1 + q\delta$ and hence $q\alpha_i^2 \leq (1+q\delta)\alpha_i$, applying Lemma 5.2 to α and to each conditional distribution $p^{(i)}$ gives, by the chain rule for relative entropy,

$$\begin{aligned}q^2\sum_{i,j=1}^q R_{ij}^2 - 1 &= \left(q\sum_{i=1}^q\alpha_i^2 - 1\right) + q\sum_{i=1}^q\alpha_i^2\left(q\sum_{j=1}^q(p^{(i)}_j)^2 - 1\right)\\ &\leq \frac{1+q\delta}{c_q}\left(D(\alpha\|u) + \sum_{i=1}^q\alpha_i D(p^{(i)}\|u)\right)\\ &= \frac{1+q\delta}{c_q}D(R\|U)\,.\end{aligned} \tag{5.7}$$

Substituting (5.7) into (5.6) produces the exponent $\theta n D(R\|U)$ with

$$\theta = \frac{d(1+q\delta)T^2}{2c_q}\,.$$

As $\delta, \rho \downarrow 0$, we have $T \to 1/(q-1)$ and $\theta \to d/[2(q-1)^2c_q] = d/d_2(q) < 1$. We therefore first fix $\delta \in (0, 1/(2q))$ small enough that

$$d(1+q\delta) < d_2(q)\,, \tag{5.8}$$

and then choose $\rho > 0$ small enough that (5.1) holds together with

$$\theta = \frac{d(1+q\delta)T^2}{2c_q} < 1\,, \qquad 4\rho \leq T^2\,. \tag{5.9}$$

Step 3: Summation over overlaps. Combining (5.6) and (5.7) and dropping the indicator $\mathbf{1}_{\{\sigma,\tau\in\mathcal{B}_n\}}$ gives, uniformly for $t \in L_{4\rho}$,

$$\mathbb{E}\big|\widehat{Z}^{\mathrm{B}}_{G_n}(t)\big|^2 \le C\mathbb{E}_R\Big[e^{\theta n D(R\|U)}\Big]\,. \tag{5.10}$$

Under independent uniform colorings σ, τ, the count matrix nR has the multinomial distribution $\mathrm{Mult}(n, U)$ on $k = q^2$ categories. Since $\theta \in (0, 1)$ by (5.9), Theorem 5.3 bounds the right-hand side of (5.10) by $C(1-\theta)^{-(q^2-1)} < \infty$. □

We fix δ and ρ as in Lemma 5.4. By Cauchy–Schwarz, $\mathbb{E}|\widehat{Z}^{\mathrm{B}}_{G_n}(t)| \le (\mathbb{E}|\widehat{Z}^{\mathrm{B}}_{G_n}(t)|^2)^{1/2}$, so combining Lemmas 5.1 and 5.4 gives

$$\sup_n \sup_{t\in L_{4\rho}} \mathbb{E}\big|\widehat{Z}_{G_n}(t)\big| < \infty\,. \tag{5.11}$$

This is the first-moment hypothesis of Proposition 6.1 below for the interval I.

6 Proof of Potts zero-freeness

Sections 4 and 5 provide the Taylor-truncation and moment inputs of Theorem 3.1. The remaining requirement is that the tube carrying the moment bound lie in the domain $\mathcal{D}$ of the compensated cycle model $\mathcal{W}$ from Lemma 3.3. Every inequality defining $\mathcal{D}$ holds strictly on an interval $I_s = [-s, 0]$ with $s < 1$, $(q-1)s \le 1$, and $ds^2 < 1$, so thin tubes around such an interval lie in $\mathcal{D}$. We record the resulting criterion for all such s and every $q \ge 2$, because it is used again for the Ising antiferromagnet in Section 7 and in Appendices B and C. For $r > 0$, let $U_r(I_s)$ and $L_r(I_s)$ denote the open and closed r-neighborhoods of I_s, so that $U_r(I_s) = U_r$ and $L_r(I_s) = L_r$ when $s = 1/(q-1)$.

Proposition 6.1 (Zero-freeness from a first-moment bound)**.** *Fix an integer $q \ge 2$ and $d > 0$. Let $0 < s < 1$ satisfy $(q-1)s \le 1$ and $ds^2 < 1$, and put $I_s = [-s, 0]$ and $x_s = (1-(q-1)s)/(1+s)$, so that $t(x_s) = -s$ and $t([x_s, 1]) = I_s$. Suppose that, for some $r > 0$,*

$$\sup_n \sup_{t\in L_r(I_s)} \mathbb{E}\big|\widehat{Z}_{G_n}(t)\big| < \infty\,. \tag{6.1}$$

Then there exists $\rho' \in (0, r/4]$ such that $\widehat{Z}_{G_n}$ converges in distribution to the function $\mathcal{W}$ of Lemma 3.3 in the locally uniform topology on $U_{4\rho'}(I_s)$, and

$$\Pr\big[\widehat{Z}_{G_n}(t) \ne 0 \text{ for every } t \in L_{\rho'}(I_s)\big] \longrightarrow 1\,. \tag{6.2}$$

Moreover, there exists $\rho_z > 0$ such that $\Pr[Z_{G_n}(q, z) \ne 0$ whenever $\mathrm{dist}(z, [x_s, 1]) \le \rho_z] \to 1$.

Proof. For $t \in I_s$ we have $|t| \le s < 1$, $(q-1)|t|^3 \le s^2 < 1$, $d|t|^2 \le ds^2 < 1$, and $\Re(1-dt) \ge 1$. Thus the compact interval I_s lies in the open set $\mathcal{D}$, and we may fix $\rho' \in (0, r/4]$ with $L_{4\rho'}(I_s) \subset \mathcal{D}$. Apply Theorem 3.1 on the connected open set $\Omega = U_{4\rho'}(I_s)$ with compact set $K = L_{\rho'}(I_s)$, expansion point $t_0 = 0$, random functions $f_n = \widehat{Z}_{G_n}$ for $n \ge n_0 = \lceil d \rceil$, and limit $f = \mathcal{W}$. Condition (i) holds by (6.1) and Lemma 3.2, since every compact subset of Ω lies in $L_{4\rho'}(I_s) \subseteq L_r(I_s)$. For condition (ii), let $k \ge 1$ and $m = \max\{k-1, 3\}$: by Lemma 4.2, the Taylor coefficients of $\widehat{Z}_{G_n}$ at $t = 0$ through degree m converge jointly in distribution to those of $\prod_{\ell=3}^{m}(1+(q-1)t^\ell)^{P_\ell}$, which coincide almost surely with those of $\mathcal{W}$ by Lemma 3.3. Condition (iii) holds because $\mathcal{W}$ is almost surely nowhere zero on $\mathcal{D} \supset K$. Theorem 3.1 now gives the convergence in distribution and (6.2).

To return to the original parameter z, recall that $Z_G(q, z) = q^{|V(G)|}b(z)^{|E(G)|}\widehat{Z}_G(t(z))$ by (2.2), where $b(x) = (x+q-1)/q > 0$ and $t(x) \in I_s$ for $x \in [x_s, 1]$. By continuity of b and t near $[x_s, 1]$ and compactness, there exists $\rho_z > 0$ such that $b(z) \ne 0$ and $t(z) \in L_{\rho'}(I_s)$ whenever $\mathrm{dist}(z, [x_s, 1]) \le \rho_z$. On the event in (6.2), $Z_{G_n}(q, z) \ne 0$ for all such z. □

Proof of Theorem 1.1. Let $s = 1/(q-1)$, so that $I_s = I$ and $x_s = 0$. Then $s \le 1/2$, $(q-1)s = 1$, and $ds^2 = d/(q-1)^2 < 1$ by $d < d_2(q)$ and (3.3). The bound (5.11) is (6.1) with $r = 4\rho$, so Proposition 6.1 gives $\rho' \in (0, \rho]$ such that (6.2) holds on $L_{\rho'}(I)$, and $\rho_z > 0$ such that $Z_{G_n}(q, z) \neq 0$ for all $z \in K_{\rho_z}$ with probability tending to one. This is Theorem 1.1 with $\rho = \rho_z$. □

7 Ising and Viana–Bray zero-freeness

We now prove the zero-freeness assertions of Theorem 1.5 by applying Proposition 6.1 (antiferromagnet) and Theorem 3.1 (spin glass) to the normalized two-state polynomial $\mathcal{I}_{G_n,J}(t)$ in (1.3). Unlike the $q \ge 3$ Potts calculation of Section 5, no balanced-window truncation is needed for two spins: both complex second-moment bounds reduce directly to the quadratic exponential moment of a sum of independent Rademacher variables. In the unsigned antiferromagnet, Hölder's inequality decouples the two single-replica magnetizations and the overlap magnetization. In the Viana–Bray spin glass, averaging over the quenched edge signs cancels the single-replica terms identically and gives the explicit bound

$$\mathbb{E}\big|\mathcal{I}_{G_n,J}(t)\big|^2 \le (1 - d|t|^2)^{-1/2} \qquad (n \ge d,\ d|t|^2 < 1)\,.$$

In the short-cycle expansion, each unsigned ℓ-cycle contributes $1 + t^\ell$, whereas random edge signs split the ℓ-cycles into unfrustrated and frustrated cycles contributing $1 + t^\ell$ and $1 - t^\ell$ with independent Poisson counts of equal mean $d^\ell/(4\ell)$, whose deterministic linear terms cancel without exponential compensation.

Proof of the zero-freeness assertions of Theorem 1.5. Step 1: Rademacher quadratic exponential moment. For a sum $S = \sum_{v=1}^n \xi_v$ of n independent uniform signs $\xi_v \in \{-1, 1\}$, introducing an independent standard normal variable $g \sim \mathcal{N}(0,1)$ and using $\mathbb{E}_g[e^{yg}] = e^{y^2/2}$, independence of the signs, and $\cosh x \le e^{x^2/2}$ for $x \in \mathbb{R}$ gives, for every $0 \le \lambda < 1$,

$$\mathbb{E}\big[e^{\lambda S^2/(2n)}\big] = \mathbb{E}_g\Big[\big(\cosh(g\sqrt{\lambda/n})\big)^n\Big] \le \mathbb{E}_g\big[e^{\lambda g^2/2}\big] = (1-\lambda)^{-1/2}\,. \tag{7.1}$$

Step 2: Antiferromagnetic Ising zero-freeness. Write $I_n = \mathcal{I}_{G_n,1}$ and let $\sigma, \tau \in \{-1,1\}^n$ be independent uniform spin configurations. Define the overlap configuration $\eta_v = \sigma_v\tau_v \in \{-1,1\}$ and the spin sums $S_\xi = \sum_{v=1}^n \xi_v$ for $\xi \in \{\sigma, \tau, \eta\}$; marginally, each S_ξ is a sum of n independent uniform signs. For $n \ge d$, taking the expectation over $G_n \sim G(n, d/n)$ edge by edge gives the exact identity

$$\mathbb{E}\big|I_n(t)\big|^2 = \mathbb{E}_{\sigma,\tau}\left[\prod_{u<v}\left\{1 + \frac{d}{n}\left(t\sigma_u\sigma_v + \bar{t}\,\tau_u\tau_v + |t|^2\eta_u\eta_v\right)\right\}\right].$$

Fix $T < \infty$ and suppose $|t| \le T$. Taking absolute values inside $\mathbb{E}_{\sigma,\tau}$ and applying $|1+w| \le \exp(\Re w + |w|^2/2)$ to each of the $\binom{n}{2}$ factors bounds the sum of the $O(n^{-2})$ quadratic terms over all pairs $u < v$ by a constant depending only on d and T. Using $\sum_{u<v}\xi_u\xi_v = (S_\xi^2 - n)/2$ for $\xi \in \{\sigma,\tau,\eta\}$, absorbing the $O(1)$ quadratic and diagonal terms $-d\Re t - d|t|^2/2$ into $C = C(d,T) \ge 1$, and bounding $\Re t$ above by $(\Re t)_+ = \max\{\Re t, 0\}$ yields

$$\mathbb{E}\big|I_n(t)\big|^2 \le C\mathbb{E}_{\sigma,\tau}\left[\exp\left\{\frac{d}{2n}\left[(\Re t)_+(S_\sigma^2 + S_\tau^2) + |t|^2 S_\eta^2\right]\right\}\right].$$

Set $\Lambda = d(|t|^2 + 2(\Re t)_+)$. Whenever $0 < \Lambda < 1$, applying (7.1) directly when $(\Re t)_+ = 0$, or Hölder's inequality with weights $d(\Re t)_+/\Lambda$, $d(\Re t)_+/\Lambda$, and $d|t|^2/\Lambda$ together with (7.1) when $(\Re t)_+ > 0$, gives

$$\mathbb{E}\big|I_n(t)\big|^2 \le C(1-\Lambda)^{-1/2} \qquad (n \ge d)\,,$$

and the same inequality holds at $\Lambda = 0$ ($t = 0$) since $I_n(0) = 1$.

Under the identification $[2] \leftrightarrow \{-1, 1\}$, $I_n(t)$ is the $q = 2$ case of $\widehat{Z}_{G_n}(t)$ in (2.1), the map $t(z) = (z-1)/(z+1)$ is the $q = 2$ case of the transformation of Section 2.1, and (1.4) is the $q = 2$ case of (2.2). Given $0 < x_* < 1$ with $d((1-x_*)/(1+x_*))^2 < 1$, put $s = -t(x_*) = (1-x_*)/(1+x_*) \in (0,1)$, so that $ds^2 < 1$ and $x_s = (1-s)/(1+s) = x_*$, and choose $r \in (0, 1-s)$ with $d((s+r)^2 + 2r) < 1$. If $\operatorname{dist}(t, [-s, 0]) \le r$, then $|t| \le s + r$ and $(\Re t)_+ \le r$, so $\Lambda \le d((s+r)^2 + 2r) < 1$ and the bound above, with $T = s + r$, is uniform on this closed tube. By Cauchy–Schwarz, $I_n = \widehat{Z}_{G_n}$ therefore satisfies (6.1) with $q = 2$, and Proposition 6.1 supplies $\rho_z > 0$ such that, with probability tending to one, $Z_{G_n}(2, z) \neq 0$ whenever $\operatorname{dist}(z, [x_*, 1]) \le \rho_z$. This is the zero-freeness assertion of Theorem 1.5(i) with $\rho = \rho_z$.

Step 3: Viana–Bray spin-glass zero-freeness. Now write $I_n = \mathcal{I}_{G_n, J}$ and $r_d = \min\{1, d^{-1/2}\}$. Averaging each two-replica edge factor $(1 + tJ_{uv}\sigma_u\sigma_v)(1 + \bar{t}J_{uv}\tau_u\tau_v)$ over the uniform sign $J_{uv} \in \{-1, 1\}$ eliminates the linear terms in J_{uv} and leaves $1 + |t|^2\eta_u\eta_v$, where $\eta_v = \sigma_v\tau_v$. Put $S = \sum_{v=1}^n \eta_v$. For $n \ge d$ and $d|t|^2 < 1$, each edge factor satisfies $1 + (d/n)|t|^2\eta_u\eta_v \ge 1 - d|t|^2/n > 0$; averaging over the graph and signs and using $1 + y \le e^y$ for $y \in \mathbb{R}$, $\sum_{u<v} \eta_u\eta_v = (S^2 - n)/2 \le S^2/2$, and (7.1) gives

$$\mathbb{E}\big|I_n(t)\big|^2 = \mathbb{E}_\eta\left[\prod_{u<v}\left(1 + \frac{d}{n}|t|^2\eta_u\eta_v\right)\right] \le \mathbb{E}_\eta\big[e^{d|t|^2S^2/(2n)}\big] \le (1 - d|t|^2)^{-1/2}\,.$$

By Cauchy–Schwarz and Lemma 3.2, $(I_n)_{n \ge n_0}$ with $n_0 = \lceil d \rceil$ satisfies condition (i) of Theorem 3.1 on the open disk $\Omega_{\mathrm{sg}} = \{t \in \mathbb{C} : |t| < r_d\}$.

In the edge-subset expansion of $I_n(t)$, leaf supports again vanish, while an ℓ-cycle C has $\prod_{uv \in E(C)} \sigma_u\sigma_v = 1$ for every spin configuration σ and therefore contributes the factor $1 + J(C)t^\ell$, where $J(C) = \prod_{e \in E(C)} J_e \in \{-1, 1\}$. Let $C_\ell^\pm$ be the number of ℓ-cycles of G_n with $J(C) = \pm 1$, and let $(P_\ell^+, P_\ell^-)_{\ell \ge 3}$ be independent with $P_\ell^\pm \sim \mathrm{Pois}(d^\ell/(4\ell))$. Fix $m \ge 3$. The proof of Lemma 4.2, with the cycle weight $q - 1$ replaced by $J(C)$, shows that with probability $1 - O_m(n^{-1})$ the cycles of length at most m in G_n are vertex-disjoint and

$$I_n(t) \equiv \prod_{\ell=3}^m (1 + t^\ell)^{C_\ell^+}(1 - t^\ell)^{C_\ell^-} \pmod{t^{m+1}}\,.$$

This event depends only on G_n, and on it distinct short cycles share no edge, so given G_n their signs are independent and uniform. Hence, for bounded f, $\mathbb{E}f((C_\ell^+, C_\ell^-)_{\ell \le m}) = \mathbb{E}\bar{f}((C_\ell(G_n))_{\ell \le m}) + O_m(\|f\|_\infty n^{-1})$, where $\bar{f}(c)$ averages f over independent uniform signs attached to c_ℓ cycles of each length ℓ. Since $\bar{f}$ is bounded on the discrete set $\{0, 1, 2, \dots\}^{m-2}$, Lemma 4.1 gives $\mathbb{E}\bar{f}((C_\ell(G_n))_\ell) \to \mathbb{E}\bar{f}((P_\ell)_\ell)$, and this limit is $\mathbb{E}f((P_\ell^+, P_\ell^-)_{\ell \le m})$ by Poisson thinning: if $P \sim \mathrm{Pois}(\lambda)$ items receive independent uniform signs, the numbers of positive and negative items have joint generating function $e^{\lambda((x+y)/2 - 1)} = e^{\lambda(x-1)/2}e^{\lambda(y-1)/2}$. As in the last paragraph of the proof of Lemma 4.2, the Taylor coefficients of $I_n(t)$ at $t = 0$ through degree m therefore converge jointly in distribution to those of

$$Q_m^{\mathrm{sg}}(t) = \prod_{\ell=3}^m (1 + t^\ell)^{P_\ell^+}(1 - t^\ell)^{P_\ell^-}\,.$$

On any closed disk $|t| \le r' < r_d$, write $P_\ell^+ \log(1 + t^\ell) + P_\ell^- \log(1 - t^\ell) = (P_\ell^+ - P_\ell^-)t^\ell + P_\ell^+[\log(1 + t^\ell) - t^\ell] + P_\ell^-[\log(1 - t^\ell) + t^\ell]$; because P_ℓ^+ and P_ℓ^- have equal mean $d^\ell/(4\ell)$, the deterministic linear terms cancel and, as in (3.5),

$$\mathbb{E}\left[\sum_{\ell=3}^\infty \sup_{|t| \le r'} \big|P_\ell^+ \log(1 + t^\ell) + P_\ell^- \log(1 - t^\ell)\big|\right] \le C\sum_{\ell=3}^\infty \left[\sqrt{\frac{d^\ell}{2\ell}}(r')^\ell + \frac{d^\ell}{2\ell}(r')^{2\ell}\right] < \infty\,.$$

Exponentiating the sum of this series, and setting the result equal to 1 on the null event where the series fails to converge locally uniformly on Ω_{sg}, defines a random holomorphic function on Ω_{sg} that is almost surely nowhere zero and whose Taylor coefficients at $t = 0$ through degree m almost surely match those of Q_m^{sg}, since the terms with $\ell > m$ are $O(t^{m+1})$, as in Step 2 of the proof of Lemma 3.3. Given $\beta_* > 0$ with $d \tanh^2 \beta_* < 1$, we have $\tanh \beta_* < r_d$; fix $r \in (\tanh \beta_*, r_d)$. Applying Theorem 3.1 on Ω_{sg} with this limit function, expansion point $t_0 = 0$, and compact set $\{|t| \le r\}$, where condition (ii) holds for each $k \ge 1$ by the above with $m = \max\{k-1, 3\}$ as in the proof of Proposition 6.1, shows that $\mathcal{I}_{G_n,J}(t) \neq 0$ on $|t| \le r$ (and in particular on $|t| \le \tanh \beta_*$) with probability tending to one. Because $\cosh \beta \ge 1$ and $|\tanh \beta| \le \tanh \beta_* < r$ on $[-\beta_*, \beta_*]$, continuity and compactness yield $\rho = \rho(d, \beta_*) > 0$ such that $\cosh \beta \neq 0$ and $|\tanh \beta| \le r$ whenever $\operatorname{dist}(\beta, [-\beta_*, \beta_*]) \le \rho$. By (1.5), on the same high-probability event, $Z^{\mathrm{VB}}_{G_n,J}(\beta) \neq 0$ throughout that closed ρ-neighborhood. □

8 Counting from zero-freeness

We count with the normalized polynomials that Sections 6 and 7 proved zero-free. In the variable t all three are polynomials (the Viana–Bray partition function is entire in β, but $\mathcal{I}_{G,J}$ is a polynomial in $t = \tanh \beta$), and their zero-free regions are tubes and disks. Given these regions, Barvinok's interpolation method [Bar16, PR17] reduces counting to computing the first $O(\log(n/\varepsilon))$ logarithmic Taylor coefficients at $t = 0$. We give this coefficient algorithm first and then the interpolation step.

Subset Möbius inversion decomposes logarithmic coefficients over connected induced subgraphs [SEHH66]. Since $G(n, d/n)$ has unbounded maximum degree, bounded-degree coefficient algorithms [LSS19, PR17, PR19, YYZ22] do not directly give polynomial running time. We enumerate connected vertex sets and compute each contribution from the configuration sum; Cayley's tree bound controls the number of these sets on G_n.

Fix $q \ge 2$. For a finite graph H with edge signs $J = (J_{ab})_{ab \in E(H)} \in \{-1, 1\}^{E(H)}$, put

$$P_{H,J}(t) = \mathbb{E}_\sigma \prod_{ab \in E(H)} \left(1 + t J_{ab} A_{\sigma_a \sigma_b}\right), \tag{8.1}$$

where σ is uniform on $[q]^{V(H)}$; here J_{ab} is the sign of the edge ab, not an entry of the all-ones matrix J of Section 2.1. With all signs equal to 1, $P_{H,J} = \widehat{Z}_H$ by (2.1). For $q = 2$, the identification $[2] \leftrightarrow \{-1, 1\}$ gives $A_{\sigma_a \sigma_b} = \sigma_a \sigma_b$, so $P_{H,J} = \mathcal{I}_{H,J}$ by (1.3). An induced subgraph keeps the signs of its edges. Since $P_{H,J}(0) = 1$, its logarithm is a formal power series without constant term, and we define the logarithmic coefficients $\ell_j(H)$, which also depend on q and J, by

$$\log P_{H,J}(t) = \sum_{j \ge 1} \ell_j(H)\, t^j . \tag{8.2}$$

Each $\ell_j(H)$ is rational. To isolate the contribution to $\ell_j(H)$ that uses every vertex of H, define the vertex Möbius weight

$$\omega_j(H) = \sum_{T \subseteq V(H)} (-1)^{|V(H)| - |T|} \ell_j(H[T]) . \tag{8.3}$$

Multiplicativity of $P_{H,J}$ over connected components makes $\omega_j(H)$ vanish on disconnected graphs. The degree-j terms involve at most j edges, so they touch at most $2j$ vertices. These two facts give the following decomposition.

Lemma 8.1 (Local decomposition)**.** *For every finite graph G with edge signs and every integer $j \ge 1$,*

$$\ell_j(G) = \sum_{\substack{S \subseteq V(G),\ 1 \le |S| \le 2j \\ G[S] \text{ connected}}} \omega_j(G[S]) . \tag{8.4}$$

Proof. By Möbius inversion on the subset lattice of $V(G)$, we have $\ell_j(G) = \sum_{S \subseteq V(G)} \omega_j(G[S])$, and the term $S = \varnothing$ vanishes because $P_{G[\varnothing],J} = 1$. It remains to show that $\omega_j(H) = 0$ if H is disconnected or if $|V(H)| > 2j$.

First suppose $H = H_1 \sqcup H_2$ with $V(H_1), V(H_2) \neq \varnothing$. Multiplicativity of $P_{H,J}$ implies $\ell_j(H[T]) = \ell_j(H_1[T \cap V(H_1)]) + \ell_j(H_2[T \cap V(H_2)])$ for every $T \subseteq V(H)$. In (8.3), summing the first term over $T \cap V(H_2) \subseteq V(H_2)$ and the second over $T \cap V(H_1) \subseteq V(H_1)$ gives zero in both cases, so $\omega_j(H) = 0$.

For the support bound, expand the edge product in (8.1): $P_{H,J}(t) = \sum_{F \subseteq E(H)} w(F)\, t^{|F|}$, where $w(F) = \mathbb{E}_\sigma \prod_{ab \in F} J_{ab} A_{\sigma_a \sigma_b}$ depends only on F and its signs, because the colors are independent and uniform. Writing $\log P_{H,J} = \sum_{i \geq 1} (-1)^{i+1} i^{-1} (P_{H,J} - 1)^i$, each contribution to the degree-j coefficient is indexed by a tuple of nonempty edge sets $F_1, \dots, F_i \subseteq E(H)$ with $\sum_{g=1}^{i} |F_g| = j$, and its weight $(-1)^{i+1} i^{-1} \prod_g w(F_g)$ does not depend on the ambient graph. The union of their endpoints $U = \bigcup_g V(F_g)$ has size at most $2j$, and the tuple occurs in $\ell_j(H[T])$ exactly when $U \subseteq T$. In (8.3), its net coefficient is therefore $\sum_{U \subseteq T \subseteq V(H)} (-1)^{|V(H)|-|T|}$, which vanishes unless $U = V(H)$. Consequently $\omega_j(H) = 0$ whenever $|V(H)| > 2j$. □

Let $N_s(G)$ denote the number of connected vertex subsets $S \subseteq V(G)$ of size s. An exponential bound on $N_s(G_n)$ in s suffices because $j \leq k = O(\log(n/\varepsilon))$ in the interpolation step.

Lemma 8.2 (Connected-set counts)**.** *For every fixed $B > 2e(d+1)$, with probability $1 - O(n^{-1})$ over $G_n \sim G(n, d/n)$, simultaneously for all integers $s \geq 1$,*

$$N_s(G_n) \leq n^2 B^s \,. \tag{8.5}$$

Proof. Every connected subgraph on s vertices contains a spanning tree. By Cayley's formula, there are s^{s-2} trees on a fixed s-vertex set, so for $s \geq 2$ we have $\mathbb{E}[N_s(G_n)] \leq \binom{n}{s} s^{s-2} (d/n)^{s-1} \leq n[e(d+1)]^s$, and the final bound also holds trivially for $s = 1$. By Markov's inequality and a union bound over $s \geq 1$, the probability that (8.5) fails for some $s \geq 1$ is at most $n^{-1} \sum_{s \geq 1} (e(d+1)/B)^s = O(n^{-1})$. □

To guarantee polynomial running time on every input graph G (not only on typical realizations of G_n), we fix an integer $B > 2e(d+1)$ and impose (8.5) as a deterministic stopping rule: the enumeration halts and reports failure as soon as the layer of connected vertex sets of size s exceeds $n^2 B^s$. By Lemma 8.2, this stopping rule is inactive on G_n with probability $1 - O(n^{-1})$.

Lemma 8.3 (Exact coefficient oracle)**.** *Fix integers $q, B \geq 2$. There exist a constant $C = C(q, B)$ and a deterministic algorithm which, given an n-vertex graph G with edge signs J and an integer $k \geq 1$, runs in time $\mathrm{poly}(n,k)C^k$, with all intermediate rationals of bit length $\mathrm{poly}(n,k)$, and either outputs the exact rational numbers $\ell_1(G), \dots, \ell_k(G)$ or reports failure. If $N_s(G) \leq n^2 B^s$ for every $s \geq 1$, the algorithm does not report failure.*

Proof. We enumerate connected vertex subsets $S \subseteq V(G)$ layer by layer for $s = 1, \dots, \min\{2k, n\}$. Layer $s = 1$ consists of the n singletons. Given layer $s-1$, we form layer s by adjoining to each accepted set S' of size $s-1$ every vertex $v \in V(G) \setminus S'$ adjacent to S', deduplicating the resulting sets and aborting immediately if the number of distinct sets in layer s exceeds $n^2 B^s$. Since every connected graph on $s \geq 2$ vertices has a spanning tree leaf whose removal leaves a connected graph on $s-1$ vertices, this breadth-first expansion finds every connected set of size s. Because each of the at most $n^2 B^{s-1}$ sets in layer $s-1$ has at most n extensions, constructing or aborting at layer s takes $\mathrm{poly}(n,k)B^{2k}$ operations.

For each accepted connected set S of size $s \le 2k$ and each subset $T \subseteq S$, enumerate all $q^{|T|}$ colorings σ of $G[T]$. Each edge factor $1+tJ_{ab}A_{\sigma_a\sigma_b}$ has the form $1+ct$ with $c \in \{\pm 1, \pm(q-1)\}$, so σ contributes $q^{-|T|}\prod_c (1+ct)^{e_c}$ to $P_{G[T],J}(t)$, where e_c is the number of edges ab of $G[T]$ with $J_{ab}A_{\sigma_a\sigma_b} = c$, and the first $k+1$ coefficients $a_0(T)=1, a_1(T), \dots, a_k(T)$ of $P_{G[T],J}$ are computed from binomial coefficients. Logarithmic differentiation of (8.2) gives the recursion

$$\ell_j(G[T]) = a_j(T) - \frac{1}{j}\sum_{r=1}^{j-1} r\,\ell_r(G[T])\,a_{j-r}(T) \qquad (1 \le j \le k).$$

This recursion takes $O(k^2)$ arithmetic operations. Since $\sum_{T\subseteq S} q^{|T|} = (q+1)^{|S|}$, computing $\omega_j(G[S])$ via (8.3) for all $1 \le j \le k$ takes $\mathrm{poly}(k)(q+1)^{|S|}$ operations per connected set S. Summing (8.4) over all accepted sets of size $s \le 2j$ yields $\ell_1(G), \dots, \ell_k(G)$ in total time $\mathrm{poly}(n,k)[B(q+1)]^{2k}$, with all intermediate rationals having bit length $\mathrm{poly}(n,k)$. Indeed, $a_r(T) \in q^{-|T|}\mathbb{Z}$ and $|a_r(T)| \le (2k^2(q-1))^r$, so by induction on j the recursion gives $\ell_j(G[T]) \in (j!\,q^{j|T|})^{-1}\mathbb{Z}$ and $|\ell_j(G[T])| \le (4k^2(q-1))^j$; each $\omega_j(G[S])$ is a signed sum of at most 4^k such numbers, and $\ell_j(G)$ is a sum of at most $2kn^2B^{2k}$ numbers $\omega_j(G[S])$. □

The interpolation step is Barvinok's method [Bar16] in the form given by Patel and Regts [PR17]. If a polynomial p with $p(0) \neq 0$ has no zeros in the open disk of radius M about 0 and $|t_1| < M$, then the exponential of the degree-k Taylor polynomial of $\log p$ at 0, evaluated at t_1, is a multiplicative ε-approximation to $p(t_1)$ in the sense of [PR17, Definition 1.1] once $k \ge \max\{1, \log(\deg(p)/\varepsilon)\}/(1-|t_1|/M)$, by the bound (8) in the proof of [PR17, Lemma 2.2] (cf. [PR17, Corollary 2.3]). For zero-free tubes around a segment, we use Barvinok's polynomial from [PR17, Lemma 4.3], denoted there by ϕ_ρ and here by ϕ_δ: for $\delta \in (0,1)$ it satisfies $\phi_\delta(0) = 0$ and $\phi_\delta(1) = 1$, and it maps a disk of radius $M_\delta > 1$ about 0 into $\{w \in \mathbb{C} : -\delta \le \Re w \le 1+2\delta,\ |\Im w| \le 2\delta\}$, whose points lie within distance $2\sqrt{2}\delta$ of $[0,1]$.

Proposition 8.4 (Counting from zero-freeness)**.** *Fix $d > 0$ and an integer $q \ge 2$, and let $G_n \sim G(n, d/n)$.*

(i) *Let $x_0 \in [0,1)$, and suppose that, for some $\varrho > 0$, with probability tending to one $\widehat{Z}_{G_n}$ has no zeros within distance ϱ of $t([x_0,1]) = [t(x_0), 0]$. Then there exist a constant $C = C(q, d, x_0, \varrho)$ and a deterministic algorithm which, given an n-vertex graph G, $x \in [x_0, 1]$, and $\varepsilon \in (0,1)$, runs in time $O((n/\varepsilon)^C)$ and outputs a number $\widetilde{Z}$ satisfying, with probability tending to one over G_n,*

$$(1-\varepsilon)Z_{G_n}(q,x) \le \widetilde{Z} \le (1+\varepsilon)Z_{G_n}(q,x)$$

simultaneously for all $x \in [x_0, 1]$ and $\varepsilon \in (0,1)$.

(ii) *Let $q = 2$, let J assign independent uniform signs to the edges of G_n, let $\beta_* > 0$, and suppose that, for some $r > \tanh\beta_*$, with probability tending to one $\mathcal{I}_{G_n,J}$ has no zeros in $\{|t| \le r\}$. Then the conclusion of* (i) *holds for $Z^{\mathrm{VB}}_{G_n,J}(\beta)$ in place of $Z_{G_n}(q,x)$, simultaneously for all $\beta \in [-\beta_*, \beta_*]$ and $\varepsilon \in (0,1)$, with $C = C(d, \beta_*, r)$, for an algorithm that also receives the signs J, and with probability tending to one over (G_n, J).*

Proof. Fix an integer $B > 2e(d+1)$, and let $\mathcal{G}_n$ be the intersection of the event (8.5), which has probability $1 - O(n^{-1})$ by Lemma 8.2, with the zero-free event of the hypothesis. Neither event depends on the target parameter or on ε. Given the input, the algorithm runs the oracle of Lemma 8.3 with $k = O(\log(n/\varepsilon))$ chosen below, which takes time $O((n/\varepsilon)^C)$ on every input, and outputs 0 if the oracle reports failure or if the real number Θ whose exponential it forms below (the value at $w = 1$ of the degree-k polynomial in (i), resp. $\sum_{j\le k}\ell_j(G)\tilde{t}^{\,j}$ with the computed approximation $\tilde{t}$ of $\tanh\beta$ in (ii)) satisfies $|\Theta| > n^3$; on $\mathcal{G}_n$ it obtains $\ell_1(G), \dots, \ell_k(G)$. Write $m = |E(G)| \le n^2$.

For (i), let $p = \widehat{Z}_G$ (all signs equal to 1), $t_1 = t(x) \in [t(x_0), 0]$, and $\delta = \min\{\varrho, 1/2\}/3$. Then $g(w) = p(t_1\phi_\delta(w))$ is a real polynomial of degree at most $m \deg(\phi_\delta)$ with $g(0) = 1$ and $g(1) = p(t_1)$. On $\mathcal{G}_n$ it has no zeros in $\{|w| \le M_\delta\}$, because $t_1\phi_\delta(w)$ lies within distance $2\sqrt{2}\delta|t_1| < \varrho$ of $[t_1, 0] \subseteq t([x_0, 1])$. Since $\phi_\delta(0) = 0$, the first k Taylor coefficients of $\log g$ at 0 are obtained by substituting $t_1\phi_\delta(w)$ into $\sum_{j\le k} \ell_j(G)t^j$ and truncating, as in the proof of [PR17, Theorem 1.2]. By the estimate above with $M = M_\delta$ at the point 1, the exponential ξ of the resulting degree-k polynomial at $w = 1$ is a multiplicative $(\varepsilon/4)$-approximation to $p(t_1)$ for some $k = O(\log(n/\varepsilon))$. All coefficients are real and $p(t_1) > 0$, so $\xi/p(t_1) \in [e^{-\varepsilon/4}, e^{\varepsilon/4}]$. By (2.2), $q^n b(x)^m \xi$ approximates $Z_G(q, x)$ within the same factor.

For (ii), let $p = \mathcal{I}_{G,J}$ and $t_1 = \tanh\beta$. On $\mathcal{G}_n$, p has no zeros in $\{|t| \le r\}$ and $|t_1| \le \tanh\beta_* < r$, so the estimate applies directly with $M = r$, and k can be chosen independently of β because $1 - |t_1|/r \ge 1 - \tanh\beta_*/r$: the exponential ξ of $\sum_{j\le k} \ell_j(G)t_1^j$ is a multiplicative $(\varepsilon/4)$-approximation to $p(t_1)$, and again $\xi/p(t_1) \in [e^{-\varepsilon/4}, e^{\varepsilon/4}]$. By (1.5), $2^n(\cosh\beta)^m\xi$ approximates $Z^{\mathrm{VB}}_{G,J}(\beta)$ within the same factor.

In both parts the algorithm outputs a number within relative error $\varepsilon/4$ of $q^n b(x)^m\xi$, resp. $2^n(\cosh\beta)^m\xi$, so its ratio to the target lies between $(1-\varepsilon/4)e^{-\varepsilon/4} \ge 1-\varepsilon$ and $(1+\varepsilon/4)e^{\varepsilon/4} \le 1+\varepsilon$. All exponentials are evaluated by truncated Taylor series after halving the argument, and $\tanh\beta = (e^{2\beta}-1)/(e^{2\beta}+1)$ and $\cosh\beta = (e^\beta + e^{-\beta})/2$ are formed from them. In (ii) the algorithm computes $u = e^{2\beta}(1+\eta)$ with $|\eta| \le \min\{1/2, \varepsilon/(16\max\{1, \sum_{j\le k} j|\ell_j(G)|\})\}$ and puts $\tilde{t} = (u-1)/(u+1)$; then $\tilde{t} = \tanh(\beta + \frac{1}{2}\log(1+\eta)) \in (-1, 1)$ and $|\tilde{t} - \tanh\beta| \le |\eta|$, which changes $\sum_{j\le k} \ell_j(G)t^j$ by at most $\varepsilon/16$. It also computes $\cosh\beta$ to relative error $\varepsilon/(16\max\{m, 1\})$, and in both parts it computes e^Θ to relative error $\varepsilon/16$; since $e^{\varepsilon/16}(1+\varepsilon/16)(1+\varepsilon/(16\max\{m,1\}))^m \le e^{3\varepsilon/16} \le 1+\varepsilon/4$ and $(1-\varepsilon/16)^3 \ge 1-\varepsilon/4$, this gives the relative error $\varepsilon/4$ used above. This takes $\mathrm{poly}(n, k, \log(1/\varepsilon))$ operations: the bound required of $|\eta|$ is at least $2^{-\mathrm{poly}(n,k)}\varepsilon$ by the bit lengths of the oracle's outputs, and $|\Theta| \le n^3$, so that $O(\log n)$ halvings and squarings suffice.

On $\mathcal{G}_n$ and for $n \ge n_0(q, x_0, \beta_*)$, the algorithm never outputs 0 because of the condition $|\Theta| > n^3$: there $|\Theta - \log p(t_1)| < 1$, and $|\log p(t_1)| = O(n^2)$ by (2.2) and (1.5), since $(1-1/q)^m \le b(x)^m \le 1$, $Z_G(q, x) \le q^n$, $Z_G(q, x) \ge q^n x_0^m$ if $x_0 > 0$, and $Z_G(q, x) \ge Z_G(q, 0) \ge 1$ if $x_0 = 0$ (the coefficients of $Z_G(q, \cdot)$ are nonnegative, and the integer $Z_G(q, 0)$ is nonzero on $\mathcal{G}_n$), while $2^n e^{-\beta_* m} \le Z^{\mathrm{VB}}_{G,J}(\beta) \le 2^n e^{\beta_* m}$ and $1 \le \cosh\beta \le e^{\beta_*}$. Time counts arithmetic operations. For rational target parameters and ε of bit length L, every number above has bit length polynomial in n, k, and L, so the number of bit operations is $O((n/\varepsilon)^C)$ times a polynomial in L; in this case we replace the parameter $1 - e^{-1/\delta}$ of ϕ_δ by a nearby rational number, which keeps $\phi_\delta(0) = 0$ and $\phi_\delta(1) = 1$ and, by continuity in that parameter and since $2\sqrt{2}\delta < \varrho$, keeps the image of the disk at distance less than ϱ from $[0, 1]$. □

Proof of Theorems 1.2 and 1.5 (counting). In the proof of Theorem 1.1, Proposition 6.1 gives (6.2) on $L_{\rho'}(I)$ for some $\rho' > 0$. Proposition 8.4(i) with $x_0 = 0$ and $\varrho = \rho'$ therefore gives Theorem 1.2, with $C = C(q, d)$. In Step 2 of Section 7, Proposition 6.1 with $q = 2$ gives (6.2) for $\widehat{Z}_{G_n} = I_n$ on a closed tube of some width $\rho' > 0$ around $t([x_*, 1])$. Proposition 8.4(i) with $x_0 = x_*$ and $\varrho = \rho'$ therefore gives the counting assertion of Theorem 1.5(i), with $C = C(d, x_*)$. In Step 3 of Section 7, $\mathcal{I}_{G_n,J}$ has no zeros in $\{|t| \le r\}$ for some $r \in (\tanh\beta_*, r_d)$ with probability tending to one. Proposition 8.4(ii) therefore gives the counting assertion of Theorem 1.5(ii), with $C = C(d, \beta_*)$. □

A Colorability consequences

Proof of Corollary 1.3. Set $a = 1/2 + \eta$ and $q_0 = ad/\log d$. Since $d_2(q) \sim 2q\log q$ as $q \to \infty$, we have $d_2(q_0)/d \to 2a > 1$ as $d \to \infty$. Direct differentiation shows that d_2 is strictly

increasing on $[3,\infty)$, so every integer $q \geq q_0$ satisfies $d < d_2(q)$ for all sufficiently large d. □

Proof of Corollary 1.4. For $q = 1$, return 1 if G is edgeless and 0 otherwise. For $q = 2$, return $2^{c(G)}$ if G is bipartite and 0 otherwise, where $c(G)$ is the number of connected components of G. For $q \geq 3$, compare our zero-freeness threshold $d_2(q)$ with the sufficient noncolorability threshold $d_{\mathrm{nc}}(q) = (2q-1)\log q$ of Achlioptas and Naor [AN05, Theorems 1 and 2]:

- if $d < d_2(q)$, apply the algorithm of Theorem 1.2 at $x = 0$;
- if $d \geq d_{\mathrm{nc}}(q)$, output 0, which is exact on the high-probability event that G_n is not q-colorable [AN05, Theorems 1 and 2].

The remaining integers $q \geq 3$ are those for which $d \in [d_2(q), d_{\mathrm{nc}}(q))$. Because $d_{\mathrm{nc}}(q)$ is increasing in q and

$$d_2(q+1) = \frac{2q^3 \log q}{q^2-1} > 2q \log q > d_{\mathrm{nc}}(q)\,,$$

the intervals $[d_2(q), d_{\mathrm{nc}}(q))$ for $q \geq 3$ are pairwise disjoint. Hence at most one integer $q_d \geq 3$ can fall into this uncovered range; if none does, q_d may be chosen arbitrarily. Here q and d are fixed before $n \to \infty$, and the constant-zero algorithm applies only at the proper-coloring endpoint $x = 0$, since $Z_G(q,x) > 0$ whenever $x > 0$. □

B Positive-temperature intervals

Proof of the extension in (1.2). In the second-moment bound of Section 5.2, the parameter interval enters the entropy comparison (5.9) only through the maximum modulus of t on the interval. On the positive-temperature subinterval $[x_0, 1] \subset (0,1]$, this maximum modulus is $s = |t(x_0)| = (1-x_0)/(q-1+x_0) < 1/(q-1)$, so the condition $d < d_2(q) = 2(q-1)^2 c_q$ is relaxed to $ds^2 < 2c_q$, which is (1.2). In the notation of Section 6, $t([x_0,1]) = I_s$ and $x_s = x_0$.

Fix $\delta \in (0, 1/(2q))$ with $d(1+q\delta)s^2 < 2c_q$, and then choose $\rho > 0$ small enough that (5.1) holds and $T = s + 4\rho$ satisfies (5.9). Section 5 uses the condition $d < d_2(q)$ only to make $\theta < 1$ in (5.9), which here holds by the choice of δ and ρ. Since $L_{4\rho}(I_s) \subseteq L_{4\rho}$, Lemma 5.1 applies on $L_{4\rho}(I_s)$. On $L_{4\rho}(I_s)$ we have $|t| \leq T$ and $\Re t \leq 4\rho \leq T^2$, which are the only properties of the tube used in Steps 1 and 2 of the proof of Lemma 5.4, so that proof applies with this T. By Cauchy–Schwarz, $\sup_n \sup_{t \in L_{4\rho}(I_s)} \mathbb{E}|\widehat{Z}_{G_n}(t)| < \infty$. Since $0 < s < 1/(q-1) \leq 1/2$ and $ds^2 < 2c_q < 1$ by (3.3), Proposition 6.1 with $r = 4\rho$ gives $\rho' \in (0, \rho]$ such that, with probability tending to one, $\widehat{Z}_{G_n}$ has no zeros in $L_{\rho'}(I_s)$, and also a fixed-width zero-free neighborhood of $[x_0, 1]$ for $Z_{G_n}(q, \cdot)$. Proposition 8.4(i) with this x_0 and $\varrho = \rho'$ then gives the claimed FPTAS simultaneously for all $x \in [x_0, 1]$ and $\varepsilon \in (0,1)$. □

C Extension to the condensation threshold

This appendix adapts the overlap-concentration strategy of Chen, Wang, and Zhang [CWZ26, Section 4] to $G(n, d/n)$ and extends Theorems 1.1 and 1.2 from $d_2(q)$ up to the Potts condensation threshold, including the proper-coloring endpoint. We obtain the overlap estimate from the planted free-energy bound of [CKPZ18, Proposition 3.6] by adding independent Poisson edges; the corresponding estimate in [CWZ26, Section 4.3] uses random pinning.

To state the threshold, fix $q \geq 3$ and recall from Section 2.1 that $b(z) = (z+q-1)/q$, $t(z) = (z-1)/(z+q-1)$, and $A = q\,\mathrm{Id} - J$. For $z \in \mathbb{C} \setminus \{-(q-1)\}$, define the $q \times q$ symmetric normalized edge matrix $M_z = J + t(z)A$, so that $M_z(a,b) = 1 + t(z)A_{ab}$ has diagonal entries $z/b(z)$, off-diagonal entries $1/b(z)$, and row sums q. Let $\mathcal{P}_{\mathbf{u}}$ denote the set of probability distributions ν on the probability simplex $\{\mu \in [0,1]^q : \sum_{a=1}^q \mu_a = 1\}$

with mean $\mathbb{E}_{\mu\sim\nu}[\mu] = \mathbf{u} = (1/q, \dots, 1/q)$. For $r > 0$, $x \in [0,1)$, and independent samples $\mu_1, \mu_2, \dots \sim \nu \in \mathcal{P}_{\mathbf{u}}$ independent of $D \sim \mathrm{Pois}(r)$, set

$$V_k = \frac{1}{q}\sum_{a=1}^{q}\prod_{i=1}^{k}(M_x\mu_i)_a\,, \qquad E_x = \mu_1^{\top} M_x \mu_2\,.$$

Because $\mathbb{E}[(M_x\mu_i)_a] = (M_x\mathbf{u})_a = 1$, we have $\mathbb{E}[V_k] = 1$ for every $k \geq 0$, $\mathbb{E}[DV_D] = r$, and $\mathbb{E}[E_x] = 1$. Subtracting $\log q + (r/2)\log b(x)$ from the Potts Bethe functional of Coja-Oghlan, Krzakala, Perkins, and Zdeborová [CKPZ18, equation (1.2)], with $c = 1 - x$, gives the normalized functional and the Erdős–Rényi Potts condensation threshold of [CKPZ18, equations (1.5) and (1.7)]:

$$\begin{aligned} \Psi_{r,x}(\nu) &= \mathbb{E}[V_D \log V_D] - \frac{r}{2}\,\mathbb{E}[E_x \log E_x]\,, \\ d_{\mathrm{cond}}(q,x) &= \inf\Big\{r > 0 : \sup_{\nu\in\mathcal{P}_{\mathbf{u}}} \Psi_{r,x}(\nu) > 0\Big\}\,, \end{aligned} \tag{C.1}$$

with $0\log 0 = 0$. At the proper-coloring endpoint $x = 0$, $d_{\mathrm{cond}}(q,0) = (2q-1)\log q - 2\log 2 + o_q(1)$ [COV13, BCOH+16]; see also the discussion following [CKPZ18, Theorem 1.2].

Theorem C.1 (Potts zero-freeness and counting up to condensation)**.** *Fix an integer $q \geq 3$, $x_* \in [0,1)$, and $0 < d < d_{\mathrm{cond}}(q, x_*)$, and let $G_n \sim G(n, d/n)$. There exists $\rho = \rho(q, d, x_*) > 0$ such that*

$$\Pr\big[Z_{G_n}(q,z) \neq 0 \text{ whenever } \operatorname{dist}(z, [x_*, 1]) \leq \rho\big] \longrightarrow 1\,.$$

Consequently, there exist a constant $C = C(q, d, x_*)$ and a deterministic algorithm which, given an n-vertex graph G, $x \in [x_*, 1]$, and $\varepsilon \in (0,1)$, runs in time $O((n/\varepsilon)^C)$ and outputs a number $\widetilde{Z}$ satisfying, with probability tending to one over G_n,

$$(1-\varepsilon)Z_{G_n}(q,x) \leq \widetilde{Z} \leq (1+\varepsilon)Z_{G_n}(q,x)$$

simultaneously for all $x \in [x_*, 1]$ and $\varepsilon \in (0,1)$.

The case $x_* = 0$ contains Theorems 1.1 and 1.2 because $d_2(q) \leq d_{\mathrm{cond}}(q,0)$. Indeed, let $d < d_2(q)$. Proposition 6.1, as applied in the proof of Theorem 1.1, gives $\widehat{Z}_{G_n} \to \mathcal{W}$ in distribution, locally uniformly on a neighborhood of $I \ni -1/(q-1)$, and $\mathcal{W}(-1/(q-1)) \neq 0$ almost surely by Lemma 3.3, since $I \subset \mathcal{D}$ by the proof of that proposition. Hence $Z_{G_n}(q,0) \geq c_0\, q^n (1-1/q)^{|E(G_n)|}$ with probability bounded below, for some $c_0 > 0$; since $|E(G_n)| \leq dn/2 + n^{2/3}$ with high probability, the second assertion of [CKPZ18, Theorem 1.2] excludes $d > d_{\mathrm{cond}}(q,0)$.

In the main text and Appendix B, the condition $d < d_2(q)$ (or (1.2)) enters only through the uniform complex first-moment bound of Section 5 (namely (5.11), or its analog on $L_{4\rho}(I_s)$ in Appendix B) and through the inequality $d\,t(x)^2 < 1$ at the left endpoint $x = 0$, resp. $x = x_0$, which it implies by (3.3). For general $d < d_{\mathrm{cond}}(q, x_*)$, the global second-moment estimate of Section 5 no longer covers the target parameter range because rare graphs with large partition function inflate $\mathbb{E}|\widehat{Z}_{G_n}(t)|^2$, yet below condensation their contribution to the first moment remains negligible. Theorem C.1 therefore reduces directly to the following first-moment estimate.

Proposition C.2 (Complex first moment below condensation)**.** *Fix an integer $q \geq 3$, $x_* \in [0,1)$, and $0 < d < d_{\mathrm{cond}}(q, x_*)$, and let $G_n \sim G(n, d/n)$. Then $d\,t(x_*)^2 < 1$, and there exists an open neighborhood $W \subset \mathbb{C} \setminus \{-(q-1)\}$ of $[x_*, 1]$ such that*

$$\sup_n \sup_{z\in W} \mathbb{E}\big|\widehat{Z}_{G_n}(t(z))\big| < \infty\,. \tag{C.2}$$

Proof of Theorem C.1 from Proposition C.2. Put $s_* = |t(x_*)|$, so that $I_{s_*} = t([x_*, 1])$ and $x_{s_*} = x_*$ in the notation of Proposition 6.1. Then $s_* > 0$ because $x_* < 1$, $(q-1)s_* \le 1$ because $x_* \ge 0$, so $s_* < 1$, and $ds_*^2 < 1$ by Proposition C.2. Since t is a Möbius transformation, $t(W)$ is an open neighborhood of I_{s_*}; hence $L_r(I_{s_*}) \subset t(W)$ for some $r > 0$, and (C.2) gives (6.1). Proposition 6.1 therefore gives $\rho' > 0$ such that, with probability tending to one, $\widehat{Z}_{G_n}$ is zero-free on $L_{\rho'}(I_{s_*})$, and $\rho_z > 0$ such that, with probability tending to one, $Z_{G_n}(q, z) \neq 0$ whenever $\operatorname{dist}(z, [x_*, 1]) \le \rho_z$; this is the first assertion of Theorem C.1 with $\rho = \rho_z$. Proposition 8.4(i) with $x_0 = x_*$ and $\varrho = \rho'$ gives the stated algorithm. □

To prove Proposition C.2, we work on ordered Poisson multigraphs and transfer to $G(n, d/n)$ by conditioning on simplicity and comparing likelihoods. For $s \ge 0$, let P_s be the law of an ordered Poisson multigraph $G = (N_{uv})_{(u,v)\in[n]^2}$ with independent edge counts $N_{uv} \sim \mathrm{Pois}(s/(2n))$ on all ordered pairs $(u, v) \in [n]^2$, including loops, so that $m = |E(G)| = \sum_{u,v} N_{uv} \sim \mathrm{Pois}(sn/2)$. For $z \in \mathbb{C} \setminus \{-(q-1)\}$, define

$$w_{G,z}(\sigma) = \prod_{(u,v)\in[n]^2} M_z(\sigma_u, \sigma_v)^{N_{uv}}, \qquad F_G(z) = q^{-n} \sum_{\sigma\in[q]^n} w_{G,z}(\sigma). \tag{C.3}$$

Whenever G is simple ($N_{uu} = 0$ for all u and $N_{uv} + N_{vu} \in \{0, 1\}$ for all $u < v$) with underlying undirected graph $\bar{G}$, symmetry of M_z gives $F_G(z) = \widehat{Z}_{\bar{G}}(t(z))$; moreover, $w_{G,\bar{z}} = \overline{w_{G,z}}$ since $M_{\bar{z}} = \overline{M_z}$. Let $\mathbb{E}_\sigma$, $\mathbb{E}_{\sigma,\tau}$, $\Pr_\sigma$, and $\Pr_{\sigma,\tau}$ denote averages and probabilities over independent uniform colorings $\sigma, \tau \in [q]^n$. Write $\alpha(\sigma)$ for the color-proportion vector of σ, call a point $x \in (0, 1]$ a *real anchor*, and at a real anchor x set $\kappa_s(\sigma) = \frac{1}{2}snt(x)q\|\alpha(\sigma) - \mathbf{u}\|_2^2 \le 0$. Poisson averaging and Jensen's inequality give

$$\mathbb{E}_{\mathsf{P}_s}[w_{G,x}(\sigma)] = e^{\kappa_s(\sigma)}, \qquad \mathbb{E}_{\mathsf{P}_s}[F_G(x)] = \mathbb{E}_\sigma[e^{\kappa_s(\sigma)}] \in [e^{-s/2}, 1], \tag{C.4}$$

since $\mathbb{E}_\sigma[q\|\alpha(\sigma) - \mathbf{u}\|_2^2] = (q-1)/n$ and $t(x)(q-1) \in (-1, 0]$.

Proposition C.2 rests on two Poisson-multigraph lemmas: the first records monotonicity and the Kesten–Stigum bound for d_{cond} and extracts the planted free-energy input below condensation from [CKPZ18], and the second converts that input into a local uniform complex first-moment bound.

Lemma C.3 (Planted free energy below condensation)**.** *Fix an integer $q \ge 3$. The threshold $x \mapsto d_{\mathrm{cond}}(q, x)$ is nondecreasing on $[0, 1)$ and satisfies $d_{\mathrm{cond}}(q, x) \le t(x)^{-2}$. Moreover, for every $x \in (0, 1)$ and $0 < s < d_{\mathrm{cond}}(q, x)$,*

$$\mathbb{E}_{\mathsf{P}_s}[F_G(x) \log F_G(x)] = o(n). \tag{C.5}$$

Lemma C.4 (Poisson complex first-moment transfer)**.** *Fix an integer $q \ge 3$, $x_* \in [0, 1)$, and $d > 0$ with $d\,t(x_*)^2 < 1$, and let $\Delta \in (0, d)$ be such that (C.5) holds at $s = d + \Delta$ for every $x \in (0, 1) \cap [x_*, 1)$. Then every $y \in [x_*, 1]$ has a bounded open neighborhood O_y with $\overline{O_y} \subset \mathbb{C} \setminus \{-(q-1)\}$ such that, for some $n_y \ge 1$,*

$$\sup_{n\ge n_y} \sup_{z\in O_y} \mathbb{E}_{\mathsf{P}_d}[e^{dm/n}|F_G(z)|] < \infty. \tag{C.6}$$

Proof of Proposition C.2 from Lemmas C.3 and C.4. By Lemma C.3, $d < d_{\mathrm{cond}}(q, x_*) \le t(x_*)^{-2}$, so $d\,t(x_*)^2 < 1$. Fix $\Delta \in (0, \min\{d, d_{\mathrm{cond}}(q, x_*) - d\})$. For every $x \in [x_*, 1)$, monotonicity in Lemma C.3 gives $d + \Delta < d_{\mathrm{cond}}(q, x)$, so (C.5) holds at $s = d + \Delta$ whenever $x \in (0, 1) \cap [x_*, 1)$. Lemma C.4 therefore supplies, at every $y \in [x_*, 1]$, a bounded open neighborhood O_y and a threshold n_y satisfying (C.6).

By compactness, choose a finite subcover $O_{y_1}, \dots, O_{y_k}$ of $[x_*, 1]$, set $W = \bigcup_{i=1}^k O_{y_i}$, and let $n_W = \max_i n_{y_i}$. Let $n > d$, and let $\mathcal{S}$ be the event that G is simple. Then

$$\Pr_{\mathsf{P}_d}[\mathcal{S}] = e^{-d/2}\big(e^{-d/n}(1+d/n)\big)^{\binom{n}{2}} \geq e^{-d/2-d^2/4}\,,$$

and conditional on $\mathcal{S}$ the underlying graph $\bar{G}$ has m edges and is distributed as $G(n, d/(n+d))$. For a simple graph Γ on $[n]$, the likelihood ratio of $G(n,d/n)$ to $G(n,d/(n+d))$ at Γ is $(1+d/n)^{|E(\Gamma)|}(1-d^2/n^2)^{\binom{n}{2}-|E(\Gamma)|} \leq e^{d|E(\Gamma)|/n}$. Since $F_G(z) = \widehat{Z}_{\bar{G}}(t(z))$ on $\mathcal{S}$, for every $z \in W$

$$\mathbb{E}|\widehat{Z}_{G_n}(t(z))| \leq \mathbb{E}_{\mathsf{P}_d}[e^{dm/n}|F_G(z)| \mid \mathcal{S}] \leq e^{d/2+d^2/4}\,\mathbb{E}_{\mathsf{P}_d}[e^{dm/n}|F_G(z)|]\,,$$

which (C.6) bounds uniformly in $z \in W$ and $n \geq n_W$. Since $\overline{W} \subset \mathbb{C} \setminus \{-(q-1)\}$ is compact, $\sup_{z\in W}|t(z)| < \infty$ absorbs the finitely many remaining n, proving (C.2). □

Section C.1 proves Lemma C.3, whose monotonicity step follows [CWZ26, proof of Lemma 4.1]. To prove Lemma C.4 in Sections C.2–C.5, we follow [CWZ26, Section 4] and separate coloring pairs by their overlap $R_{\sigma,\tau}$ from (5.3). For z near $[x_*, 1]$, pairs with nearly uniform overlap have an explicit complex mean, bounded under the Kesten–Stigum condition $d\,t(x_*)^2 < 1$ (Lemma C.5). At a real anchor x, the free energy at density $d+\Delta$, tested against an added layer of edges of density Δ, makes the other pairs, and the graphs with exponentially large partition function, exponentially rare (Lemma C.7). One weighted Cauchy–Schwarz inequality combines the two near each anchor (Section C.5). The base graph is always a Poisson multigraph of the fixed density d, where these averages are explicit, and the same inequality absorbs the factor $e^{dm/n}$ in (C.6). Tracking the single-edge log-partition increment $\log(e/x)$ as $x \downarrow 0$ lets one small positive anchor cover the origin, where monochromatic edges have weight 0.

C.1 Planted free energy below condensation

Proof of Lemma C.3. For a probability vector μ we have $A\mu = q(\mu - \mathbf{u})$, hence

$$M_x\mu = q\mathbf{u} + q\,t(x)(\mu - \mathbf{u})\,, \qquad \mu_1^\top M_x \mu_2 = 1 + q\,t(x)\langle \mu_1 - \mathbf{u}, \mu_2 - \mathbf{u}\rangle\,. \tag{C.7}$$

Monotonicity. Let $0 \leq y \leq x < 1$, so that $t(y) \leq t(x) < 0$, and put $\alpha_0 = t(x)/t(y) \in (0,1]$. For $\nu \in \mathcal{P}_{\mathbf{u}}$, let $\nu' \in \mathcal{P}_{\mathbf{u}}$ be the law of $\mathbf{u} + \alpha_0(\mu - \mathbf{u})$ for $\mu \sim \nu$. By (C.7), the pair (y, ν') produces the same variables V_k as (x,ν) and the edge variable $E' = 1 + \alpha_0(E_x - 1)$. Convexity of $v \mapsto v\log v$ gives $\mathbb{E}[E'\log E'] \leq \alpha_0\mathbb{E}[E_x \log E_x] \leq \mathbb{E}[E_x \log E_x]$, where the last step uses $\mathbb{E}[E_x \log E_x] \geq \mathbb{E}[E_x]\log\mathbb{E}[E_x] = 0$. Hence $\Psi_{r,y}(\nu') \geq \Psi_{r,x}(\nu)$ for every $r > 0$, and $d_{\mathrm{cond}}(q,y) \leq d_{\mathrm{cond}}(q,x)$. This is the argument of [CWZ26, proof of Lemma 4.1, equation (19)].

Kesten–Stigum bound. We show $d_{\mathrm{cond}}(q,x) \leq t(x)^{-2}$ (cf. [COEJ+18, equation (1.7)]); the expansion below follows [COEJ+18, Lemma 5.4], cf. [CWZ26, equation (18)]. Let $x \in [0,1)$, $t = t(x)$, and $s > t^{-2}$. For $\epsilon \in (0,1)$, let $\nu_\epsilon \in \mathcal{P}_{\mathbf{u}}$ be the law of $\mathbf{u} + \epsilon(e_a - \mathbf{u})$ with a uniform on $[q]$, and write $a_1, a_2, \dots$ for the independent colors of $\mu_1, \mu_2, \dots$. By (C.7), $(M_x\mu_i)_b = 1 + \epsilon t A_{ba_i}$ and $E_x = 1 + \epsilon^2 t A_{a_1a_2}$. Expanding the product in V_k and using $\sum_b A_{ba} = 0$ and $A^2 = qA$ gives $V_k = 1 + \epsilon^2 t^2 \sum_{1\leq i<j\leq k} A_{a_ia_j} + O_k(\epsilon^3)$, whose summands are pairwise uncorrelated with mean 0 and second moment $\mathbb{E}[A^2_{a_1a_2}] = q-1$. Since $\mathbb{E}[V_k] = \mathbb{E}[E_x] = 1$, we may replace $v\log v$ in (C.1) by $v\log v - v + 1$, which is nonnegative and equals $\frac{1}{2}(v-1)^2 + O(|v-1|^3)$. Discarding the event $\{D > k_0\}$ for a fixed integer $k_0 \geq 2$ then gives

$$\Psi_{s,x}(\nu_\epsilon) \geq \frac{(q-1)\epsilon^4}{2}\Big[t^4\,\mathbb{E}\Big[\tbinom{D}{2}\mathbf{1}_{\{D\leq k_0\}}\Big] - \frac{s\,t^2}{2}\Big] + O_{k_0}(\epsilon^5)\,.$$

Since $\mathbb{E}\binom{D}{2} = s^2/2$ and $st^2 > 1$, the bracket is positive for large k_0, so $\Psi_{s,x}(\nu_\epsilon) > 0$ for small ϵ. Hence $d_{\mathrm{cond}}(q,x) \leq s$ for every $s > t(x)^{-2}$.

Free energy. Fix $x \in (0,1)$ and $0 < s < d_{\mathrm{cond}}(q,x)$, and write $t = t(x)$ and $F = F_G(x)$. We apply [CKPZ18, Proposition 3.6], which assumes SYM and BAL, to the factor-graph model of [CKPZ18, Section 2] with spins $[q]$, arity 2, and the single weight function M_x, which takes values in $(0, 3/2] \subset (0,2)$. Its normalization constant [CKPZ18, equation (2.2)] equals 1; SYM [CKPZ18, Figure 1] holds because all row and column sums of M_x equal q; and BAL holds because $\mu \mapsto \mu^\top M_x \mu = 1 + t(q\|\mu\|_2^2 - 1)$ is concave and maximal at $\mathbf{u}$ by (C.7). With $m \sim \mathrm{Pois}(sn/2)$ constraints on independent uniform ordered pairs, the null model [CKPZ18, Definition 2.5] is P_s, with partition function $q^n F$. In the Bethe functional [CKPZ18, equation (2.3)], whose Poisson degree variable plays the role of D, symmetry of M_x turns each inner sum into a coordinate $(M_x\mu)_a$; with $\Lambda(v) = v \log v$, the two terms become $\frac{1}{q}\Lambda(qV_D)$ and $\frac{s}{2}\Lambda(E_x)$, and $\mathbb{E}V_D = 1$ shows that the functional equals $\log q + \Psi_{s,x}(\nu)$ at $\nu \in \mathcal{P}_*^2([q]) = \mathcal{P}_{\mathbf{u}}$. The reweighted model [CKPZ18, equation (3.4)] reweights by $q^n F$ separately for each m. Hence, with

$$\bar{a}_m = \mathbb{E}_{\mathsf{P}_s}[F \mid m]\,, \qquad \bar{c}_m = \frac{\mathbb{E}_{\mathsf{P}_s}[F \log F \mid m]}{\bar{a}_m}\,,$$

[CKPZ18, Proposition 3.6] reads $\limsup_n n^{-1}\mathbb{E}\bar{c}_m \le \sup_\nu \Psi_{s,x}(\nu) \le 0$, the last step because $s < d_{\mathrm{cond}}(q,x)$. Given m, the edges are independent and uniform, so Jensen's inequality and $\mathbb{E}_\sigma[q\|\alpha(\sigma) - \mathbf{u}\|_2^2] = (q-1)/n$ give

$$\bar{a}_m = \mathbb{E}_\sigma\big(1 + tq\|\alpha(\sigma) - \mathbf{u}\|_2^2\big)^m \in \big[(1-1/n)^m, 1\big]\,,$$

and Jensen's inequality also gives $\bar{c}_m \ge \log \bar{a}_m$. Therefore $\bar{a}_m\bar{c}_m \le \bar{c}_m - \log\bar{a}_m \le \bar{c}_m + 2m/n$ for $n \ge 2$, and $\mathbb{E}_{\mathsf{P}_s}[F\log F] = \mathbb{E}[\bar{a}_m\bar{c}_m] \le \mathbb{E}\bar{c}_m + s$, so $\limsup_n n^{-1}\mathbb{E}_{\mathsf{P}_s}[F\log F] \le 0$. Conversely, Jensen's inequality and $v\log v \ge -e^{-1}$ give $\mathbb{E}_{\mathsf{P}_s}[F\log F] \ge \mathbb{E}_{\mathsf{P}_s}[F]\log\mathbb{E}_{\mathsf{P}_s}[F] \ge -e^{-1}$, which proves (C.5). □

C.2 Poisson averages

For colorings σ, τ, let $\alpha = \alpha(\sigma)$ and $\beta = \alpha(\tau)$ be their color-proportion vectors, $R = R_{\sigma,\tau}$ their overlap matrix (5.3), and $\bar{R} = \mathbf{u}\mathbf{u}^\top$. At a real anchor x we write $t = t(x) \in (-1/(q-1), 0]$, $F = F_G(x) > 0$, and $\mu_G(\sigma) = q^{-n} w_{G,x}(\sigma)/F$, and we let P_s^σ and $\mathsf{P}_s^{\sigma,\tau}$ be the laws of independent Poisson edge counts N_{uv}, $(u,v) \in [n]^2$, with means $sM_x(\sigma_u,\sigma_v)/(2n)$ and $sM_x(\sigma_u,\sigma_v)M_x(\tau_u,\tau_v)/(2n)$; these laws, μ_G, and the quantities H, Y, and Q below depend on the anchor.

A Poisson variable N with mean ϖ satisfies $\mathbb{E}\, y^N = e^{\varpi(y-1)}$ for $y \in \mathbb{C}$, and for $y \ge 0$ the y^N-tilt of its law is $\mathrm{Pois}(\varpi y)$. Applying this to every edge count, with $\sum_{u,v} A_{\sigma_u\sigma_v} = n^2 q\|\alpha(\sigma) - \mathbf{u}\|_2^2$ and $n^{-2}\sum_{u,v} f(\sigma_u,\tau_u,\sigma_v,\tau_v) = \sum_{a,b,a',b'} R_{ab}R_{a'b'} f(a,b,a',b')$ (sums over all n^2 ordered pairs), gives at a real anchor, for $\Phi \ge 0$ and $z, z' \in \mathbb{C} \setminus \{-(q-1)\}$,

$$\mathbb{E}_{\mathsf{P}_s}\big[w_{G,x}(\sigma)\Phi(G)\big] = e^{\kappa_s(\sigma)}\,\mathbb{E}_{\mathsf{P}_s^\sigma}\Phi\,, \tag{C.8}$$

$$\mathbb{E}_{\mathsf{P}_s}\big[w_{G,z}(\sigma)w_{G,z'}(\tau)\big] = e^{n\phi^s_{z,z'}(R)}\,, \qquad \phi^s_{z,z'}(R) = \tfrac{s}{2}\big(\langle R, (M_z \otimes M_{z'})R\rangle - 1\big)\,, \tag{C.9}$$

$$\mathbb{E}_{\mathsf{P}_s}\big[w_{G,x}(\sigma)w_{G,x}(\tau)\Phi(G)\big] = e^{n\phi^s_{x,x}(R)}\,\mathbb{E}_{\mathsf{P}_s^{\sigma,\tau}}\Phi\,, \tag{C.10}$$

where $\langle R, (M_z \otimes M_{z'})R\rangle = \sum_{a,b,a',b'} R_{ab}R_{a'b'}M_z(a,a')M_{z'}(b,b')$; by linearity, (C.8) and (C.10) extend to real Φ whose absolute value is integrable on the right. In particular, (C.8) with $\Phi = 1$ and $\kappa_s \le 0$ give

$$\mathbb{E}_{\mathsf{P}_s}\big[F\ \Pr_{\mu_G}[\mathcal{A}]\big] = \mathbb{E}_\sigma\big[e^{\kappa_s(\sigma)}\mathbf{1}_{\mathcal{A}}(\sigma)\big] \le \Pr_\sigma[\mathcal{A}] \qquad (\mathcal{A} \subseteq [q]^n)\,. \tag{C.11}$$

C.3 Local pair sums

From now on, q, x_*, d, and Δ are as in Lemma C.4. Since $d\,t(x_*)^2 < 1$, we may fix once and for all

$$\begin{gathered} |t(x_*)| < t_* < 1\,, \qquad d\,t_*^2 < 1\,, \\ 0 < \eta \le 1\,, \qquad \theta_* := d\,t_*^2(1+q^2\eta) < 1\,, \qquad p := \frac{2}{1+\theta_*}\,. \end{gathered} \tag{C.12}$$

Then $p > 1$ and $p\theta_* < 1$. Let $\mathcal{V}$ be the set of $z \in \mathbb{C}\setminus\{-(q-1)\}$ with $|t(z)| < t_*$ and $\Re t(z) < t_*^2$. It is open and contains $[x_*, 1]$; since $t_* < 1$, while $t(z) \to 1$ as $|z| \to \infty$ and $|t(z)| \to \infty$ as $z \to -(q-1)$, it is bounded and its closure does not contain $-(q-1)$. The constants C_0, C_1, c_1, c_2, and c_3 introduced below depend only on q, d, x_*, Δ, and (C.12).

For $z \in \mathbb{C} \setminus \{-(q-1)\}$, define the local pair sum

$$H_G(z) = q^{-2n} \sum_{\sigma,\tau:\ \|R_{\sigma,\tau}-\bar{R}\|_{\mathrm{F}}<\eta} w_{G,z}(\sigma)\,\overline{w_{G,z}(\tau)}\,.$$

It is real, since exchanging σ and τ conjugates the summand. At a real anchor, $H = H_G(x) \in [0, F^2]$, and the pairs outside the window carry the F-weighted mass

$$Y = \frac{F^2 - H}{F} = F\,\Pr_{\mu_G^{\otimes 2}}\big[\|R_{\sigma,\tau} - \bar{R}\|_{\mathrm{F}} \ge \eta\big] \in [0, F]\,.$$

If $\|R - \bar{R}\|_{\mathrm{F}} < \eta$, the entries of $\bar{R} + u(R-\bar{R})$ for $0 \le u < 1$ are positive and at most $q^{-2} + \eta$, so integrating the second derivative of $u \mapsto D(\bar{R} + u(R-\bar{R})\|\bar{R})$ twice gives

$$D(R\|\bar{R}) \ge \frac{q^2}{2(1+q^2\eta)}\,\|R - \bar{R}\|_{\mathrm{F}}^2\,. \tag{C.13}$$

Lemma C.5 (Local pair sums)**.** *For every $z \in \mathcal{V}$, every $p' \in [1, p]$, and all $n \ge 1$, with $\phi_z = \phi_{z,\bar{z}}^d$,*

$$\mathbb{E}_{\sigma,\tau}\Big[\mathbf{1}_{\{\|R-\bar{R}\|_{\mathrm{F}}<\eta\}}\, e^{p' n \Re\phi_z(R)}\Big] \le (1 - p'\theta_*)^{-(q^2-1)}\,. \tag{C.14}$$

Consequently, with $C_0 = (1-\theta_*)^{-(q^2-1)}$ and $C_1 = (1-p\theta_*)^{-(q^2-1)/p}$, the following hold under P_d for all $n \ge 1$.

(i) *$|\mathbb{E}H_G(z)| \le C_0$ for every $z \in \mathcal{V}$.*

(ii) *At every real anchor $x \in [x_*, 1]$, $\mathbb{E}[H\mathbf{1}_{\mathcal{E}}] \le C_1\,\Pr_{\mathsf{Q}}[\mathcal{E}]^{1-1/p}$ for every set $\mathcal{E}$ of multigraphs, where Q is the law of (σ, τ, G) with σ, τ independent and uniform and G distributed as $\mathsf{P}_d^{\sigma,\tau}$ given (σ, τ), and $\Pr_{\mathsf{Q}}[\mathcal{E}] = \Pr_{\mathsf{Q}}[G \in \mathcal{E}]$.*

Proof. By Section C.2, $\langle R, (M_z \otimes M_{\bar{z}})R\rangle = \mathbb{E}_{u,v}[(1+t(z)A_{\sigma_u\sigma_v})(1+\overline{t(z)}A_{\tau_u\tau_v})]$ for independent uniform vertices $u, v \in [n]$. Since $\Re t(z) < t_*^2$ and $|t(z)|^2 < t_*^2$, the computation following (5.5) in Section 5.2, with $T = t_*$, gives $\Re\phi_z(R) \le \frac{d}{2}t_*^2(q^2\sum_{a,b}R_{ab}^2 - 1) = \frac{d}{2}t_*^2q^2\|R-\bar{R}\|_{\mathrm{F}}^2$, which is at most $\theta_* D(R\|\bar{R})$ on the window by (C.13). Since $nR \sim \mathrm{Mult}(n, \bar{R})$ on q^2 cells under independent uniform colorings, Theorem 5.3, applied with the exponent $p'\theta_*$ in place of its s, gives (C.14). By (C.9), $\mathbb{E}_{\mathsf{P}_d}H_G(z) = \mathbb{E}_{\sigma,\tau}[\mathbf{1}_{\{\|R-\bar{R}\|_{\mathrm{F}}<\eta\}}e^{n\phi_z(R)}]$, so (C.14) with $p' = 1$ gives (i). For (ii), (C.10) with $\Phi = \mathbf{1}_{\mathcal{E}}$ gives $\mathbb{E}_{\mathsf{P}_d}[H\mathbf{1}_{\mathcal{E}}] = \mathbb{E}_{\mathsf{Q}}[\mathbf{1}_{\{\|R-\bar{R}\|_{\mathrm{F}}<\eta\}}e^{n\phi_x(R)}\mathbf{1}_{\mathcal{E}}(G)]$; apply Hölder's inequality under Q with exponents p and $p/(p-1)$, and (C.14) with $p' = p$ at $z = x \in \mathcal{V}$. □

C.4 Estimates at a real anchor

At a real anchor x, the free energy below condensation controls what the local pair sums cannot see: the mass Y of pairs with correlated overlap, and the H-weighted mass of graphs with exponentially large F. Since $b(x) \in [2/3, 1]$, the entries $x/b(x)$ and $1/b(x)$ of M_x lie in $[x, 3/2]$; adding an edge multiplies F by a μ_G-average of them, so it changes $\log F$ by at most $\log(e/x)$, which is unbounded as $x \downarrow 0$. Both estimates therefore use concentration for multigraphs whose edge counts are Poisson given independent vertex labels, with explicit dependence on the single-edge influence B.

Lemma C.6 (Conditional-Poisson concentration)**.** *Let $\xi = (\xi_1, \dots, \xi_n)$ have independent coordinates in a finite set, and given ξ let $G = (N_{uv})_{(u,v)\in[n]^2}$ have independent $\mathrm{Pois}(K(\xi_u, \xi_v)/(2n))$ counts, where $0 \le K \le K_0$. Let $B > 0$, and let f satisfy $|f(G + e) - f(G)| \le B$ for every multigraph G and every added edge e, loops included. Then, with $c_f = \mathbb{E} f(G)$ and $C_\star = 1 + K_0 + K_0^2$,*

$$\mathbb{E}\big[e^{h(f(G)-c_f)}\big] \le e^{C_\star h^2 B^2 n} \quad (|h| \le 1/B)\,, \qquad \mathbb{E}\big|f(G) - c_f\big| \le 2e^{C_\star} B\sqrt{n}\,. \tag{C.15}$$

Proof. Since $|f(G) - f(\varnothing)| \le Bm$ and, given ξ, m is Poisson with mean at most $K_0 n/2$, all exponential moments below are finite. If X' is an independent copy of X and $|g(y) - g(y')| \le \omega(y, y')$ for a symmetric $\omega \ge 0$, then Jensen's inequality and symmetry give

$$\mathbb{E} e^{h(g(X)-\mathbb{E}g(X))} \le \mathbb{E}\cosh\big(h(g(X) - g(X'))\big) \le \mathbb{E}\cosh\big(h\,\omega(X, X')\big)\,.$$

For $X \sim \mathrm{Pois}(\varpi)$, $\omega = B|y - y'|$, and $|hB| \le 1$, the right-hand side is $e^{2\varpi(\cosh(hB)-1)} \le e^{2\varpi h^2 B^2}$; if $\omega \le K_0 B$, it is at most $e^{h^2 K_0^2 B^2/2}$. Given ξ, the Doob martingale of f over the n^2 edge counts has increments $g(N_{uv}) - \mathbb{E}g(N_{uv})$ with $|g(k+1) - g(k)| \le B$, and the means sum to at most $K_0 n/2$, so $\mathbb{E}[e^{h(f-\mathbb{E}[f|\xi])} \mid \xi] \le e^{K_0 h^2 B^2 n}$. Changing one ξ_v changes the means of the $2n - 1$ counts at v by at most $K_0/(2n)$ each; coupling these counts by superposition shows that $\mathbb{E}[f \mid \xi]$ moves by at most $K_0 B$, and the Doob martingale over $\xi_1, \dots, \xi_n$ gives $\mathbb{E} e^{h(\mathbb{E}[f|\xi]-c_f)} \le e^{K_0^2 h^2 B^2 n/2}$. Multiplying proves the first bound in (C.15); the second follows at $h = \pm(B\sqrt{n})^{-1}$ from $|y| \le e^y + e^{-y}$. □

Lemma C.7 (Estimates at a real anchor)**.** *There are constants $c_1 > 0$ and $c_2 \in (0, 1]$ with the following property. For every real anchor $x \in [x_*, 1]$, there are $\gamma_x \in (0, 1]$ and n_x such that, under P_d and for all $n \ge n_x$,*

$$\mathbb{E} Y \le e^{-\gamma_x n}\,, \tag{C.16}$$

$$\mathbb{E}\big[H \mathbf{1}_{\{F > e^{\gamma_x n/4}\}}\big] \le C_1 \exp\Big(-\frac{c_2 \gamma_x^2 n}{\log^2(e/x)}\Big)\,. \tag{C.17}$$

Moreover, $\gamma_x \ge c_1/\log^2(e/x)$ whenever $x \le 1/2$.

Proof. If $x = 1$, then $F \equiv 1$, so the left-hand side of (C.17) is 0, and μ_G is uniform, so $Y = \Pr_{\sigma,\tau}[\|R - \bar{R}\|_{\mathrm{F}} \ge \eta]$. This event forces $|R_{ab} - q^{-2}| \ge \eta/q$ for one of the q^2 cells, and Hoeffding's inequality [Hoe63, Theorem 2] for the cell counts $nR_{ab} \sim \mathrm{Bin}(n, q^{-2})$ gives $\mathbb{E} Y \le 2q^2 e^{-2\eta^2 n/q^2}$, which is (C.16) with $\gamma_1 = \eta^2/q^2$ for large n. Let now $x < 1$, and write $\ell = \log(e/x)$ and $s' = d + \Delta$, so that (C.5) holds at $s = s'$ by the hypothesis of Lemma C.4.

Step 1: Free energy at density s'. Let $\bar{f}_n = \mathbb{E}_\sigma \mathbb{E}_{\mathsf{P}^\sigma_{s'}} \log F$. Lemma C.6 applies to $f = \log F$ with $\xi = \sigma$, kernel $s' M_x \le \frac{3}{2} s'$ (so $K_0 = \frac{3}{2} s'$), and $B = \ell$; its constant $C'_\star = 1 + \frac{3}{2} s' + \frac{9}{4} s'^2$ does not depend on x. By (C.8) with $\Phi = \log F$ (note $|\log F| \le \ell m$),

$$\begin{aligned} \mathbb{E}_{\mathsf{P}_{s'}}[F]\,\bar{f}_n - \mathbb{E}_{\mathsf{P}_{s'}}[F \log F] &= \mathbb{E}_\sigma\big[e^{\kappa_{s'}(\sigma)}(\bar{f}_n - \mathbb{E}_{\mathsf{P}^\sigma_{s'}} \log F)\big] \\ &\le \mathbb{E}_\sigma \mathbb{E}_{\mathsf{P}^\sigma_{s'}} |\log F - \bar{f}_n| \le 2e^{C'_\star} \ell \sqrt{n}\,. \end{aligned}$$

Since $\mathbb{E}_{\mathsf{P}_{s'}} F \ge e^{-s'/2}$ by (C.4), this gives $\bar{f}_n^+ \le e^{s'/2}(\mathbb{E}_{\mathsf{P}_{s'}}[F \log F] + 2e^{C'_\star \ell\sqrt{n}})^+$, so (C.5) at $s = s'$ yields $\limsup_n \bar{f}_n/n \le 0$. By (C.8) with $\Phi = F^h$, $e^{\kappa_{s'}} \le 1$, and (C.15),

$$\mathbb{E}_{\mathsf{P}_{s'}} F^{1+h} \le \mathbb{E}_\sigma \mathbb{E}_{\mathsf{P}^\sigma_{s'}} e^{h \log F} \le e^{h\bar{f}_n + C'_\star h^2 \ell^2 n} \qquad (0 < h \le 1/\ell)\,. \tag{C.18}$$

Step 2: A test on added edges. For a coloring τ with proportions β and a multigraph $G' = (N'_{uv})$, set $S_\tau(G') = tq \sum_{(u,v)\in[n]^2} N'_{uv} \langle e_{\tau_u} - \beta, e_{\tau_v} - \beta\rangle$. Its coefficients have modulus at most $2q|t| \le 3$ and sum to $tq\|\sum_v (e_{\tau_v} - \beta)\|_2^2 = 0$, so Poisson averaging and $e^y - 1 - y \le \frac{e}{2} y^2$ for $|y| \le 1$ give the first bound in (C.20) below, with $C_{\mathrm{t}} = \frac{9e}{4}\Delta$. Since $M_x(a,b) = 1 + tq\langle e_a - \mathbf{u}, e_b - \mathbf{u}\rangle$ and $\sum_v (e_{\sigma_v} - \mathbf{u}) \otimes (e_{\tau_v} - \beta) = n(R - \alpha\beta^\top)$,

$$\mathbb{E}_{\mathsf{P}^\sigma_\Delta} S_\tau = \frac{\Delta n t^2 q^2}{2} \|R - \alpha\beta^\top\|_{\mathrm{F}}^2\,, \tag{C.19}$$

and Jensen's inequality and (C.19) give the second bound, where $\psi_x = \Delta t^2 q^2 \eta^2/16$ and $\mathcal{C}$ is the set of pairs (σ,τ) with $\|R_{\sigma,\tau} - \alpha\beta^\top\|_{\mathrm{F}} \ge \eta/2$:

$$\sup_\tau \mathbb{E}_{\mathsf{P}_\Delta} e^{\lambda S_\tau} \le e^{C_{\mathrm{t}}\lambda^2 n} \quad (0 \le \lambda \le \tfrac{1}{3})\,, \qquad \mathbb{E}_{\mathsf{P}^\sigma_\Delta} e^{\lambda S_\tau} \ge e^{2\lambda\psi_x n} \quad (\lambda \ge 0,\ (\sigma,\tau) \in \mathcal{C})\,. \tag{C.20}$$

Step 3: Correlated pairs. Let $G \sim \mathsf{P}_d$ and $G' \sim \mathsf{P}_\Delta$ be independent, so that $G + G' \sim \mathsf{P}_{s'}$ and $w_{G+G',x} = w_{G,x} w_{G',x}$, and put

$$\Xi = \mathbb{E}_{\mathsf{P}_d}\Big[F \sum_{(\sigma,\tau)\in\mathcal{C}} \mu_G(\sigma)\mu_G(\tau)\, e^{\kappa_\Delta(\sigma)}\Big]\,.$$

Let $0 < h \le 1/\ell$, $0 < \lambda \le \frac{1}{3}$, and $\lambda' = \lambda h/(1+h)$. For $(\sigma,\tau) \in \mathcal{C}$, (C.20) and (C.8) for G' give $e^{\kappa_\Delta(\sigma) + 2\lambda'\psi_x n} \le \mathbb{E}_{G'}[w_{G',x}(\sigma) e^{\lambda' S_\tau(G')}]$; summing over all pairs gives the first inequality below. Hölder's inequality with exponents $1+h$ and $(1+h)/h$, Jensen's inequality $(\sum_\tau \mu_G(\tau) e^{\lambda' S_\tau})^{(1+h)/h} \le \sum_\tau \mu_G(\tau) e^{\lambda S_\tau}$, (C.18), and (C.20) give the second:

$$\begin{aligned} e^{2\lambda'\psi_x n}\,\Xi &\le \mathbb{E}_{G,G'}\Big[F_{G+G'}(x) \sum_\tau \mu_G(\tau) e^{\lambda' S_\tau(G')}\Big] \\ &\le \exp\Big\{\frac{h}{1+h}\,[\bar{f}_n + C'_\star h \ell^2 n + C_{\mathrm{t}}\lambda^2 n]\Big\}\,. \end{aligned} \tag{C.21}$$

Choose $\lambda = \min\{\frac{1}{3}, \frac{\psi_x}{2C_{\mathrm{t}}}\}$ and $h = \min\{\frac{1}{\ell}, \frac{\lambda\psi_x}{2C'_\star \ell^2}\}$, so that the last two terms in the bracket are each at most $\frac{1}{2}\lambda\psi_x n$; for large n, also $\bar{f}_n \le \frac{1}{2}\lambda\psi_x n$. Then (C.21) and $h \le 1$ give $\Xi \le e^{-h\lambda\psi_x n/4}$. By Cauchy–Schwarz, (C.8), and $\kappa_d - \kappa_\Delta = \kappa_{d-\Delta} \le 0$,

$$\big(\mathbb{E}_{\mathsf{P}_d}[F \Pr_{\mu_G^{\otimes 2}}[\mathcal{C}]]\big)^2 \le \Xi \cdot \mathbb{E}_{\mathsf{P}_d}\Big[F \sum_\sigma \mu_G(\sigma) e^{-\kappa_\Delta(\sigma)}\Big] = \Xi \cdot \mathbb{E}_\sigma e^{\kappa_{d-\Delta}(\sigma)} \le \Xi\,.$$

Step 4: Proof of (C.16). If $\|R - \bar{R}\|_{\mathrm{F}} \ge \eta$, then $(\sigma,\tau) \in \mathcal{C}$ or one of $\|\alpha - \mathbf{u}\|_2$ and $\|\beta - \mathbf{u}\|_2$ is at least $\eta/4$, because $\|\alpha\beta^\top - \bar{R}\|_{\mathrm{F}} \le \|\alpha - \mathbf{u}\|_2 + \|\beta - \mathbf{u}\|_2$. By (C.11) and Hoeffding's inequality for the q binomial color counts, each of the last two events contributes at most $\Pr_\sigma[\|\alpha(\sigma) - \mathbf{u}\|_2 \ge \eta/4] \le 2qe^{-\eta^2 n/(8q)}$ to $\mathbb{E}Y$. With Step 3, for large n,

$$\mathbb{E}Y \le 4qe^{-\eta^2 n/(8q)} + e^{-h\lambda\psi_x n/8} \le e^{-\gamma_x n}\,, \qquad \gamma_x = \frac{1}{2}\min\Big\{\frac{\eta^2}{8q}, \frac{h\lambda\psi_x}{8}\Big\} \le 1\,.$$

If $x \le 1/2$, then $|t| \ge 1/(2q-1)$; since $C'_\star$ and C_{t} do not depend on x, this gives $\psi_x \ge \psi_0$, $\lambda \ge \lambda_0$, and $h \ge h_0/\ell^2$, where

$$\psi_0 = \frac{\Delta q^2 \eta^2}{16(2q-1)^2}\,, \qquad \lambda_0 = \min\Big\{\frac{1}{3}, \frac{\psi_0}{2C_{\mathrm{t}}}\Big\}\,, \qquad h_0 = \min\Big\{1, \frac{\lambda_0\psi_0}{2C'_\star}\Big\}\,,$$

hence $\gamma_x \ge c_1/\ell^2$ with $c_1 = \frac{1}{2}\min\{\frac{\eta^2}{8q}, \frac{h_0\lambda_0\psi_0}{8}\}$.

Step 5: Proof of (C.17). Let $\underline{a} = e^{-d/2}/2$ and $A_0 = 2C_0/\underline{a}$. By (C.4) and (C.16), $\mathbb{E}[H/F] = \mathbb{E}F - \mathbb{E}Y \ge \underline{a}$ for large n. Since $H/F \le \sqrt{H}$, $H/F \le H/A_0$ on $\{F > A_0\}$, and $\mathbb{E}H \le C_0$ by Lemma C.5(i),

$$\underline{a} \le \mathbb{E}\Big[\frac{H}{F}\mathbf{1}_{\{F\le A_0\}}\Big] + \frac{C_0}{A_0} \le \sqrt{\mathbb{E}[H\mathbf{1}_{\{F\le A_0\}}]} + \frac{\underline{a}}{2}\,,$$

so Lemma C.5(ii) gives $\Pr_{\mathsf{Q}}[F \le A_0] \ge c_{\mathsf{Q}} := (\underline{a}^2/(4C_1))^{p/(p-1)}$. Under Q, the labels $\xi_v = (\sigma_v, \tau_v)$ are independent and uniform, and given them the kernel is $dM_x(a,a')M_x(b,b') \le \frac{9}{4}d$; so Lemma C.6 applies to $\log F$ with $K_0 = \frac{9}{4}d$ and $B = \ell$, and its constant $C_{\mathsf{Q}} = 1 + \frac{9}{4}d + \frac{81}{16}d^2$ does not depend on x. With $\bar{f}_{\mathsf{Q}} = \mathbb{E}_{\mathsf{Q}}\log F$, integrating $\bar{f}_{\mathsf{Q}} - \log A_0 \le |\log F - \bar{f}_{\mathsf{Q}}|$ over $\{F \le A_0\}$ gives $\bar{f}_{\mathsf{Q}} \le \log A_0 + 2e^{C_{\mathsf{Q}}}\ell\sqrt{n}/c_{\mathsf{Q}}$, which is at most $\gamma_x n/8$ for large n. With $h' = \gamma_x/(16C_{\mathsf{Q}}\ell^2) \le 1/\ell$, (C.15) gives

$$\Pr_{\mathsf{Q}}[F > e^{\gamma_x n/4}] \le \Pr_{\mathsf{Q}}[\log F - \bar{f}_{\mathsf{Q}} > \gamma_x n/8] \le e^{-h'\gamma_x n/8 + C_{\mathsf{Q}}h'^2\ell^2 n} = e^{-\gamma_x^2 n/(256C_{\mathsf{Q}}\ell^2)}\,,$$

and Lemma C.5(ii) yields (C.17) with $c_2 = (1-1/p)/(256C_{\mathsf{Q}}) \in (0,1]$. □

C.5 Complex first moment near an anchor

If $|M_z(a,a')| \le e^{\zeta}M_x(a,a')$ for all a,a', comparing complex with real weights costs a factor $e^{\zeta m}$, which is exponential in n; the exponential rates of Lemma C.7 absorb it, which is why those rates are needed. The weighted Cauchy–Schwarz inequality below also absorbs, at bounded cost, the factor $e^{dm/n}$ in (C.6), which bounds the likelihood ratio between $G(n,d/n)$ and the Poisson multigraph conditioned to be simple.

Proof of Lemma C.4. Step 1: The bound near an anchor. Let $x \in [x_*, 1]$ be a real anchor, write $\gamma = \gamma_x$ and $\ell = \log(e/x)$, and put $\zeta = \zeta_x = \min\{\frac{1}{4}, c_2\gamma^2/(18d\ell^2)\}$. We show that there is n'_x such that, under P_d,

$$\mathbb{E}[e^{dm/n}|F_G(z)|] \le e^{d^2}\sqrt{2C_0+4} \tag{C.22}$$

for all $n \ge n'_x$ and all $z \in \mathcal{V}$ with $|M_z(a,a')| \le e^{\zeta}M_x(a,a')$ for all $a,a' \in [q]$. If $x \le 1/2$, then $\gamma \ge c_1/\ell^2$ by Lemma C.7, so $\zeta_x \ge c_3/\ell^6$ with $c_3 = \min\{\frac{1}{4}, c_2c_1^2/(18d)\}$.

Comparing the products over the m edges gives $|w_{G,z}(\sigma)| \le e^{\zeta m}w_{G,x}(\sigma)$; since $|F_G(z)|^2 - H_G(z)$ is the sum over the pairs outside the window, $|H_G(z)| \le e^{2\zeta m}H$ and $||F_G(z)|^2 - H_G(z)| \le e^{2\zeta m}(F^2 - H)$. Let $\Upsilon = \max\{1, e^{-\gamma n/4}F\}$. Then $0 \le 1 - 1/\Upsilon \le \mathbf{1}_{\{F>e^{\gamma n/4}\}}$ and $1/\Upsilon \le e^{\gamma n/4}/F$; writing $|F_G(z)|^2/\Upsilon = H_G(z) - (1-1/\Upsilon)H_G(z) + (|F_G(z)|^2 - H_G(z))/\Upsilon$ and using Lemma C.5(i), we get

$$\mathbb{E}\frac{|F_G(z)|^2}{\Upsilon} \le C_0 + \mathbb{E}[e^{2\zeta m}H\mathbf{1}_{\{F>e^{\gamma n/4}\}}] + e^{\gamma n/4}\,\mathbb{E}[e^{2\zeta m}Y]\,. \tag{C.23}$$

To remove the factors $e^{\chi m}$, average over the edge counts with the extra factor e^{χ} per edge, as in Section C.2: since $\alpha(\sigma)^{\top}M_x\alpha(\sigma) = 1 + tq\|\alpha(\sigma) - \mathbf{u}\|_2^2 \le 1$ and $\langle R, (M_x \otimes M_x)R\rangle \le \frac{9}{4}$, for every $\chi \ge 0$,

$$\mathbb{E}[e^{\chi m}F] \le \mathbb{E}[e^{\chi m}] = e^{(e^{\chi}-1)dn/2}\,, \qquad \mathbb{E}[e^{\chi m}H] \le e^{9(e^{\chi}-1)dn/8}\,\mathbb{E}H \le C_0\,e^{9(e^{\chi}-1)dn/8}\,.$$

Cauchy–Schwarz, $Y \le F$, (C.16), (C.17), and $e^{4\zeta} - 1 \le 8\zeta$ then give

$$e^{\gamma n/4}\mathbb{E}[e^{2\zeta m}Y] \le e^{\gamma n/4}(\mathbb{E}[e^{4\zeta m}F])^{1/2}(\mathbb{E}Y)^{1/2} \le e^{(\gamma/4 + 2d\zeta - \gamma/2)n} \le e^{-\gamma n/8}\,,$$
$$\mathbb{E}[e^{2\zeta m}H\mathbf{1}_{\{F>e^{\gamma n/4}\}}] \le (\mathbb{E}[e^{4\zeta m}H])^{1/2}(\mathbb{E}[H\mathbf{1}_{\{F>e^{\gamma n/4}\}}])^{1/2} \le (C_0C_1)^{1/2}e^{(9d\zeta/2 - c_2\gamma^2/(2\ell^2))n}\,,$$

because $2d\zeta \le \gamma/8$ and $\frac{9}{2}d\zeta \le \frac{1}{4}c_2\gamma^2/\ell^2$ by the choice of ζ (recall $\gamma \le 1 \le \ell$ and $c_2 \le 1$). For large n both right-hand sides are at most 1, so the right-hand side of (C.23) is at most $C_0 + 2$. Finally, for $n \ge 2d$ the same averaging with $\chi = 2d/n \le 1$ gives $\mathbb{E}[e^{2dm/n}\Upsilon] \le \mathbb{E}[e^{2dm/n}] + \mathbb{E}[e^{2dm/n}F] \le 2e^{2d^2}$, and Cauchy–Schwarz yields

$$\mathbb{E}[e^{dm/n}|F_G(z)|] \le (\mathbb{E}[e^{2dm/n}\Upsilon])^{1/2}\Big(\mathbb{E}\frac{|F_G(z)|^2}{\Upsilon}\Big)^{1/2} \le e^{d^2}\sqrt{2C_0+4}\,.$$

Step 2: Neighborhoods. Let $y \in [x_*, 1]$. If $y > 0$, then y is a real anchor, and the set O_y of all $z \in \mathcal{V}$ with $|M_z(a,a')| < e^{\zeta_y} M_y(a,a')$ for all a, a' is open and contains y; by (C.22), (C.6) holds on it with $n_y = n'_y$. If $y = 0$, then $x_* = 0$ and $0 \in \mathcal{V}$, and we follow the positive-anchor continuation of [CWZ26, Lemma 4.8, Steps 3–4]. For $x \in (0, 1/2]$ and $|z| \le x$, the bounds $|z| \le x$ (for the diagonal entries) and $|b(z)| \ge (q-1-x)/q$ give $|M_z(a,a')| \le \frac{b(x)}{|b(z)|}M_x(a,a') \le \frac{q-1+x}{q-1-x}M_x(a,a') \le e^{2x}M_x(a,a')$. Since $x\log^6(e/x) \to 0$ as $x \downarrow 0$, we may fix a real anchor $\hat{x} \in (0, 1/2]$ with $\{|z| \le \hat{x}\} \subset \mathcal{V}$ and $2\hat{x} \le c_3/\log^6(e/\hat{x}) \le \zeta_{\hat{x}}$; then (C.22) at the anchor $\hat{x}$ gives (C.6) on $O_0 = \{|z| < \hat{x}\}$ with $n_y = n'_{\hat{x}}$. In both cases $O_y \subseteq \mathcal{V}$, so O_y is bounded and $\overline{O_y} \subset \mathbb{C} \setminus \{-(q-1)\}$. □

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