

An LP Algorithm for Counting Eulerian Orientations Through the Lens of Quasi-polymorphism

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Abstract

The counting weighted Eulerian orientation problem $\#EO$ is a central obstruction in the classification of complex-valued Boolean Holant problems. A recent result [MWX25a] gave an FP^{NP} versus $\#P$ -hard dichotomy for $\#EO$. Its tractable side is described by quasi-polymorphisms, namely one-sided ternary XOR polymorphism conditions, together with membership in one of two tractable pairing classes.

We remove the NP oracle from the tractable side. For every signature set satisfying these conditions, we give a deterministic polynomial-time algorithm. The algorithm solves a natural linear program, extracts at each vertex a local candidate set, and shows that the LP constraints upgrade the quasi-polymorphism condition into full ternary XOR polymorphism on these candidate sets. This yields affine local structure, from which the instance reduces to a tractable Boolean $\#CSP$. Hence $\#EO$ satisfies a genuine FP versus $\#P$ -hard dichotomy.

1 Introduction

Counting problems have been studied through several robust frameworks in computational complexity. Two classical examples are counting weighted graph homomorphism problems ($\#GH$) and counting constraint satisfaction problems ($\#CSP$). Given a fixed weighted graph H , an instance of $\#GH$ consists of an input graph G ; the task is to compute the total weight of all homomorphisms from G to H . Given a fixed constraint language Γ , an instance of $\#CSP$ consists of variables and constraints, where each constraint applies a function from Γ to a scope of variables; the task is to compute the sum, over all assignments to the variables, of the product of the constraint evaluations. Although these frameworks have different syntactic forms, both evaluate sum-of-products quantities. In statistical physics, such quantities are called partition functions. A central goal is to classify which fixed families of local functions give polynomial-time computable partition functions and which give $\#P$ -hard problems. The computational complexity of these two frameworks is now fully classified by the dichotomy theorems [CCL13, CLX14].

The Holant [CLX09] framework provides a more general language for such partition functions. The full dichotomy for complex-valued Holant problems is still open. Both $\#GH$ and $\#CSP$ can be expressed in this framework through standard encodings. Holant also captures problems whose tractability is not naturally explained from the standard $\#CSP$ viewpoint, such as perfect matchings and other matchgate-related partition functions. Since Valiant’s theory of holographic

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algorithms [Val04], a long line of work has obtained dichotomy theorems for Holant problems, including Holant* [CLX11], symmetric Holant [CGW16], Holant^c [Bac21], and real Holant [SC20].

In this paper, we study the counting complex-weighted Eulerian orientation problems, denoted by #EO. Informally, a #EO instance Ω consists of a graph $G = (V, E)$ and a local EO signature f_v at each vertex v . Its value is the sum of the weights of all Eulerian orientations of G :

$$\#EO(\Omega) = \sum_{\sigma \in EO(G)} \prod_{v \in V(G)} f_v(\sigma|_{E(v)}).$$

The product is the weight of the orientation σ , obtained by multiplying the local contributions at all vertices. This problem has two intertwined origins. Combinatorially, it extends the classical problem of counting Eulerian orientations [MW92]. Physically, it also generalizes the six-vertex model, the classical ice model of Eulerian orientations on the square lattice [Lie67, Bax82]. From the viewpoint of Holant complexity, #EO became important through the study of Holant problems: #EO is a generalization of six-vertex model [CFX18] and has emerged as a key bottleneck on the path toward a full classification of complex-valued Holant problems [CFS20, SC20].

Recently, Meng, Wang, and Xia [MWX25a] established an explicit FP^{NP} versus #P-hard dichotomy for #EO. Their theorem shows that every $\#EO(\mathcal{F})$ is either #P-hard or computable in polynomial time with access to an NP oracle. The tractable side is characterized by two structural conditions: the signature set admits a common XOR₃-quasi-polymorphism (all up or all down), and belongs to one of two tractable classes, denoted $EO^{\mathcal{A}}$ and $EO^{\mathcal{P}}$. Their polynomial-time algorithm for the tractable side uses an oracle for support identification.

1.1 Our results

In this paper, we give a deterministic polynomial-time algorithm for the tractable classes of #EO identified by [MWX25a]. Our algorithm solves a natural linear program and extracts from it, at each vertex v , a local candidate set U_v . The key step is to prove that the LP constraints upgrade the quasi-polymorphism condition to a full polymorphism condition on each U_v , forcing U_v to be an affine subspace. This affine structure then allows us to choose compatible local pairings and reduce the whole instance to an ordinary Boolean #CSP instance over \mathcal{A} or \mathcal{P} , which can be evaluated in polynomial time.

Consequently, every case previously placed on the FP^{NP} side is in fact computable in deterministic polynomial time. Combining this algorithm with the hardness theorem yields a genuine FP versus #P-hard dichotomy for #EO.

Theorem 1.1 (Main dichotomy). *Let \mathcal{F} be a finite set of complex-weighted EO signatures. If both of the following conditions hold:*

- (i) \mathcal{F} admits a common XOR₃-quasi-polymorphism, i.e., every signature in \mathcal{F} admits an up XOR₃-quasi-polymorphism, or every signature admits a down XOR₃-quasi-polymorphism;
- (ii) $\mathcal{F} \subseteq EO^{\mathcal{A}}$ or $\mathcal{F} \subseteq EO^{\mathcal{P}}$,

then $\#EO(\mathcal{F}) \in FP$. Otherwise, $\#EO(\mathcal{F})$ is #P-hard.

This also removes the remaining oracle obstacle in the recent odd-arity Holant dichotomy of Meng, Wang, Xia, and Zheng [MWXZ25]. Their FP^{NP} term arises from the then-known FP^{NP} versus #P-hard dichotomy for #EO, and they explicitly observe that an ordinary FP versus #P-hard dichotomy for #EO would immediately upgrade their theorem to an FP versus #P-hard dichotomy. Theorem 1.1 provides exactly this missing ingredient. (A signature is *non-trivial* if it is not identically zero.)

Theorem 1.2 (Holant dichotomy for odd arity). *Let \mathcal{F} be a finite set of complex-valued signatures containing a non-trivial signature of odd arity. Then $\text{Holant}(\mathcal{F}) \in \text{FP}$ or $\text{Holant}(\mathcal{F})$ is $\#P$ -hard.*

Proof. We use the dichotomy theorem of Meng, Wang, Xia, and Zheng [MWXZ25] as a black box. They prove that if \mathcal{F} contains a non-trivial signature of odd arity, then $\text{Holant}(\mathcal{F})$ is either $\#P$ -hard or computable in FP^{NP} . Moreover, they explicitly state that the appearance of the NP oracle in their theorem comes solely from the then-available FP^{NP} versus $\#P$ -hard dichotomy for $\#EO$; consequently, if a standard FP versus $\#P$ -hard dichotomy for $\#EO$ is established, their Holant odd-arity dichotomy immediately becomes an FP versus $\#P$ -hard dichotomy.

Theorem 1.1 provides exactly such an FP versus $\#P$ -hard dichotomy for $\#EO$. Therefore every non-hard case in the odd-arity Holant dichotomy of [MWXZ25] is in FP. \square

2 Preliminaries

2.1 Definitions and notations

A Boolean variable is a variable taking values in the domain $\{0, 1\}$. We identify this domain with the field \mathbb{F}_2 : addition is denoted by \oplus and is the usual bit XOR, while multiplication is denoted by \wedge and is the usual bit AND. Unless explicitly stated otherwise, all variables in this paper are Boolean; for brevity, we simply write “variable” instead of “Boolean variable”.

A signature of arity r is a function $f : \{0, 1\}^r \rightarrow \mathbb{C}$. We write $\text{Var}(f) = (x_1, \dots, x_r)$ for the variables of f in their specified order. For an input $\alpha = (\alpha_1, \dots, \alpha_r) \in \{0, 1\}^r$, the value of f on α is denoted by $f(\alpha)$. We also write $\alpha(x_i) = \alpha_i$ for the value assigned by α to the variable x_i .

We often write an input α as a bit string $\alpha_1 \cdots \alpha_r$. In this notation, the i -th bit of the string is the value assigned to the i -th variable x_i of f . Thus the bit-string notation is always understood with respect to the given order of the variables.

The support of f , denoted by $\text{supp}(f)$, is the set of all strings on which f takes non-zero values $\text{supp}(f) = \{\alpha \in \{0, 1\}^r : f(\alpha) \neq 0\}$. For a signature f and a set $U \subseteq \{0, 1\}^{\text{arity}(f)}$, we write $f|_U$ for the signature obtained by restricting the support of f to U , namely $(f|_U)(\alpha) = f(\alpha)$ if $\alpha \in U$ and $(f|_U)(\alpha) = 0$ otherwise.

We denote by $=_2$ and \neq_2 the binary equality and disequality signatures, respectively: $=_2$ is 1 on 00, 11 and 0 otherwise, while \neq_2 is 1 on 01, 10 and 0 otherwise.

For a Boolean string α , let $\ell(\alpha)$ denote its length. We define $\text{HW}^= := \{\alpha \in \{0, 1\}^* : \text{wt}(\alpha) = \ell(\alpha)/2\}$, $\text{HW}^> := \{\alpha \in \{0, 1\}^* : \text{wt}(\alpha) > \ell(\alpha)/2\}$, and $\text{HW}^< := \{\alpha \in \{0, 1\}^* : \text{wt}(\alpha) < \ell(\alpha)/2\}$. An Eulerian orientation (EO) signature is a signature of arity $2d$ whose support is contained in $\text{HW}^=$.

If \mathcal{Q} is a property of signatures, we say that a signature set \mathcal{F} has property \mathcal{Q} if every $f \in \mathcal{F}$ has property \mathcal{Q} .

Unless otherwise specified, all graphs in this paper are undirected. Let $G = (V, E)$ be an undirected graph. An orientation of G assigns a direction to each edge. Under our convention, if an edge is directed from u to v , then the endpoint at the tail u receives value 1, and the endpoint at the head v receives value 0.

An orientation is called Eulerian if, at every vertex, the number of incoming edges equals the number of outgoing edges. Equivalently, under the above endpoint-value convention, every vertex is incident to the same number of endpoints assigned 0 and endpoints assigned 1. We denote by $\text{EO}(G)$ the set of all Eulerian orientations of G .

Definition 2.1 (#CSP [CC17]). A counting constraint satisfaction problem is parametrized by a set of signatures \mathcal{F} , it is denoted by $\#CSP(\mathcal{F})$. An instance of $\#CSP(\mathcal{F})$ is a finite set of variables $\{x_1, x_2, \dots, x_n\}$ and a finite set C of constraints. Each constraint is a pair consisting of a signature $f \in \mathcal{F}$ of arity k and an ordered scope $(x_{i_1}, x_{i_2}, \dots, x_{i_k})$. The output is

$$\sum_{x_1, x_2, \dots, x_n \in \{0,1\}} \prod_{(f, x_{i_1}, x_{i_2}, \dots, x_{i_k}) \in C} f(x_{i_1}, x_{i_2}, \dots, x_{i_k}).$$

Definition 2.2 (Holant [CC17]). A Holant problem is parametrized by a set of signatures \mathcal{F} , it is denoted by $\text{Holant}(\mathcal{F})$. An instance of $\text{Holant}(\mathcal{F})$ is a signature grid $\Omega = (G, \pi)$ over \mathcal{F} , which consists of a graph $G = (V, E)$ and a mapping π that assigns to each vertex $v \in V$ an $f_v \in \mathcal{F}$ and a linear order of the incident edges at v . The output is

$$\sum_{\sigma: E(G) \rightarrow \{0,1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}),$$

where σ is an assignment of $E(G)$ and $\sigma|_{E(v)}$ is the assignment σ restricted to the set $E(v)$ of incident edges of v .

The bipartite Holant problem $\text{Holant}(\mathcal{F} \mid \mathcal{G})$ is the restriction of Holant to bipartite signature grids. An instance consists of a bipartite graph $H = (U, V, E)$, where vertices in U are assigned signatures from \mathcal{F} and vertices in V are assigned signatures from \mathcal{G} .

Definition 2.3 (#EO). A counting complex-weighted Eulerian orientation problem is parametrized by a set of EO signatures \mathcal{F} , it is denoted by $\#EO(\mathcal{F})$. An instance of $\#EO(\mathcal{F})$ is a signature grid $\Omega = (G, \pi)$ over \mathcal{F} , which consists of a graph $G = (V, E)$ and a mapping π that assigns to each vertex $v \in V$ an $f_v \in \mathcal{F}$ and a linear order of the incident edges at v . The output is

$$\sum_{\sigma \in \text{EO}(G)} \prod_{v \in V} f_v(\sigma|_{E(v)}),$$

where σ is an Eulerian orientation of G and $\sigma|_{E(v)}$ is the orientation σ restricted to the set $E(v)$ of incident edges of v .

Throughout the paper, the signature set \mathcal{F} is fixed and finite. Signatures have algebraic complex values; arithmetic in the number field generated by their values is performed exactly. Consequently, all arities and support sizes of signatures appearing in instances of $\#EO(\mathcal{F})$ are constants depending only on \mathcal{F} . We therefore measure the size of an instance $I = \Omega(G, \pi)$ by $|G|$, the size of its underlying graph.

We use \leq_T and \equiv_T to denote polynomial-time Turing reductions and polynomial-time Turing equivalence, respectively.

Lemma 2.4 ([CLX09]). For every finite signature set \mathcal{F} , $\text{Holant}(\mathcal{F}) \equiv_T \text{Holant}(=_2 \mid \mathcal{F})$.

Lemma 2.5 ([CFS20]). For every finite set \mathcal{F} of EO signatures, $\#EO(\mathcal{F}) \equiv_T \text{Holant}(\neq_2 \mid \mathcal{F})$.

Lemma 2.6 ([SC20]). Let Z be the standard holographic transformation that maps \neq_2 to $=_2$. Then for every finite signature set \mathcal{F} , $\text{Holant}(\neq_2 \mid \mathcal{F}) \equiv_T \text{Holant}(Z^{-1}\mathcal{F})$.

Lemma 2.7 ([CFS20]). For every finite set \mathcal{F} of EO signatures, $\#CSP(\mathcal{F}) \equiv_T \#EO(\pi(\mathcal{F}))$. Here π maps a d -ary signature f to a $2d$ -ary EO signature by replacing each variable x_i with a complementary pair (x_i, y_i) satisfying $y_i = 1 - x_i$.

We use the standard notion of polymorphisms of relations from the algebraic approach to constraint satisfaction [JCG97, BJK05].

Definition 2.8 (Polymorphism). *Let $R \subseteq \{0, 1\}^r$ be a Boolean relation, and let $p : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean operation. For $\alpha^{(1)}, \dots, \alpha^{(k)} \in \{0, 1\}^r$, we write*

$$p(\alpha^{(1)}, \dots, \alpha^{(k)}) := (p(\alpha_1^{(1)}, \dots, \alpha_1^{(k)}), \dots, p(\alpha_r^{(1)}, \dots, \alpha_r^{(k)}))$$

for the coordinatewise application of p . We say that p is a polymorphism of R if, for all $\alpha^{(1)}, \dots, \alpha^{(k)} \in R$,

$$p(\alpha^{(1)}, \dots, \alpha^{(k)}) \in R.$$

For a signature f , we say that p is a polymorphism of f if p is a polymorphism of $\text{supp}(f)$.

Let $\text{XOR}_3 : \{0, 1\}^3 \rightarrow \{0, 1\}$ denote the ternary XOR operation, defined by $\text{XOR}_3(a, b, c) = a \oplus b \oplus c$. Thus, for tuples $\alpha, \beta, \gamma \in \{0, 1\}^r$, its coordinatewise application is

$$\text{XOR}_3(\alpha, \beta, \gamma) = \alpha \oplus \beta \oplus \gamma = (\alpha_1 \oplus \beta_1 \oplus \gamma_1, \dots, \alpha_r \oplus \beta_r \oplus \gamma_r).$$

In particular, XOR_3 is a polymorphism of a signature f if and only if, for all $\alpha, \beta, \gamma \in \text{supp}(f)$,

$$\alpha \oplus \beta \oplus \gamma \in \text{supp}(f).$$

A Boolean relation is called *affine* if it is definable by a system of linear equations over \mathbb{F}_2 . Hence a signature f whose support is closed under coordinatewise ternary XOR has affine support. To avoid confusion with the weighted affine signatures used in the Holant dichotomy, we call such a signature *support-affine*.

Definition 2.9 (XOR₃-quasi-polymorphisms for EO signatures). *Let f be an EO signature of arity $2d$.*

We say that f admits XOR₃ as a polymorphism if, for all $\alpha, \beta, \gamma \in \text{supp}(f)$,

$$\alpha \oplus \beta \oplus \gamma \in \text{supp}(f).$$

Equivalently, $\text{supp}(f)$ is closed under the minority operation; we call such a signature support-affine.

We say that f admits an up XOR₃-quasi-polymorphism if, for all $\alpha, \beta, \gamma \in \text{supp}(f)$,

$$\alpha \oplus \beta \oplus \gamma \in \text{supp}(f) \cup \text{HW}^>.$$

We say that f admits a down XOR₃-quasi-polymorphism if, for all $\alpha, \beta, \gamma \in \text{supp}(f)$,

$$\alpha \oplus \beta \oplus \gamma \in \text{supp}(f) \cup \text{HW}^<.$$

These are one-sided relaxations of the full XOR₃ polymorphism condition.

For a set \mathcal{F} of EO signatures, we say that \mathcal{F} admits a common XOR₃-quasi-polymorphism if either every $f \in \mathcal{F}$ admits an up XOR₃-quasi-polymorphism, or every $f \in \mathcal{F}$ admits a down XOR₃-quasi-polymorphism.

Remark 2.10. *The up and down XOR₃-quasi-polymorphism conditions were introduced by Meng, Wang, and Xia [MWX25a], where they are denoted by $\forall 3\uparrow$ and $\forall 3\downarrow$, respectively.*

2.2 Known results

We recall two dichotomy theorems that will be used in this paper: dichotomy for #CSP and the FP^{NP} versus #P-hard dichotomy for #EO.

Definition 2.11 (The classes \mathcal{A} and \mathcal{P}). We use \mathcal{A} and \mathcal{P} for the two standard tractable classes in the #CSP dichotomy of [CLX14]. A signature $f : \{0, 1\}^r \rightarrow \mathbb{C}$ belongs to \mathcal{A} if either $f \equiv 0$, or there exist a nonzero constant $\lambda \in \mathbb{C}$, a matrix A over \mathbb{F}_2 , and 0/1-valued affine linear forms ℓ_1, \dots, ℓ_m over \mathbb{F}_2 such that, for $X = (x_1, \dots, x_r, 1)^T$,

$$f(x_1, \dots, x_r) = \lambda \chi_{AX=0}(x_1, \dots, x_r) i^{\ell_1(X) + \dots + \ell_m(X)}.$$

Here $AX = 0$ is over \mathbb{F}_2 , while the exponent $\ell_1(X) + \dots + \ell_m(X)$ is an ordinary integer sum, equivalently read modulo 4. The class \mathcal{P} consists of signatures expressible as products of unary signatures, binary equality signatures $=_2$, and binary disequality signatures \neq_2 .

The class \mathcal{A} is a weighted tractable class for #CSP and should not be confused with the class of support-affine signatures defined above.

Theorem 2.12 (#CSP dichotomy [CLX14]). Let \mathcal{F} be a finite set of complex-valued Boolean signatures. If $\mathcal{F} \subseteq \mathcal{A}$ or $\mathcal{F} \subseteq \mathcal{P}$, then $\#\text{CSP}(\mathcal{F})$ is computable in polynomial time. Otherwise, $\#\text{CSP}(\mathcal{F})$ is #P-hard.

Theorem 2.13 (#EO dichotomy [MWX25a]). Let \mathcal{F} be a finite set of complex-valued EO signatures. If \mathcal{F} admits a common XOR₃-quasi-polymorphism and $\mathcal{F} \subseteq \text{EO}^{\mathcal{A}}$ or $\mathcal{F} \subseteq \text{EO}^{\mathcal{P}}$, then $\#\text{EO}(\mathcal{F}) \in \text{FP}^{\text{NP}}$. Otherwise, $\#\text{EO}(\mathcal{F})$ is #P-hard.

2.3 The pairing classes $\text{EO}^{\mathcal{A}}$ and $\text{EO}^{\mathcal{P}}$

Let f be an EO signature of arity $2d$. A *pairing* of $\text{Var}(f)$ is a partition of the variables of f into d unordered pairs. For a pairing P , define

$$\text{EOP}(P) := \{\tau \in \{0, 1\}^{\text{Var}(f)} : \tau(p) \oplus \tau(q) = 1 \text{ for every } \{p, q\} \in P\}.$$

Notice that $\text{EOP}(P) \subseteq \text{HW}^=$.

Definition 2.14 (Pairwise opposite signature). Let f be an EO signature of arity $2d$. We say that f is *pairwise opposite* if there exists a pairing P of $\text{Var}(f)$ such that $\text{supp}(f) \subseteq \text{EOP}(P)$.

Lemma 2.15 ([CFS20]). Every support-affine EO signature is pairwise opposite. Equivalently, if f is a support-affine EO signature, then there exists a pairing P of $\text{Var}(f)$ such that $\text{supp}(f) \subseteq \text{EOP}(P)$.

An *oriented pairing* is a list

$$\vec{P} = ((p_1^0, p_1^1), \dots, (p_d^0, p_d^1))$$

whose unordered pairs form a pairing of $\text{Var}(f)$. The order of this list gives the variable order of the induced d -ary signature $f_{\vec{P}}$. For $x = (x_1, \dots, x_d) \in \{0, 1\}^d$, let $\tau_x \in \text{EOP}(P)$ be the assignment defined by

$$\tau_x(p_j^0) = x_j, \quad \tau_x(p_j^1) = 1 - x_j \quad \text{for } j = 1, \dots, d.$$

We define $f_{\vec{P}}(x_1, \dots, x_d) := f(\tau_x)$.

Following [MWX25a], we write $f \in \text{EO}^{\mathcal{A}}$ if $f_{\vec{P}} \in \mathcal{A}$ for every oriented pairing \vec{P} , and $f \in \text{EO}^{\mathcal{P}}$ if $f_{\vec{P}} \in \mathcal{P}$ for every oriented pairing \vec{P} .

3 The LP Algorithm

Let $I = \Omega(G, \pi)$ be an instance of $\#\text{EO}(\mathcal{F})$ and $Z(I)$ be the output. We regard each incidence of an edge with a vertex as a port of that vertex. Thus every edge has exactly two ports, one at each endpoint. Under any Eulerian orientation $\sigma \in \text{EO}(G)$, the two ports of every edge receive complementary bits.

For each vertex v , suppose f_v has arity $2d_v$, and write

$$\text{supp}(f_v) = \{\alpha_{v,1}, \dots, \alpha_{v,m_v}\}.$$

For a port h of v , define

$$S_{v,h} := \{i \in [m_v] : \alpha_{v,i}(h) = 1\}.$$

Introduce nonnegative variables $\lambda_{v,1}, \dots, \lambda_{v,m_v}$ with

$$\sum_{i=1}^{m_v} \lambda_{v,i} = 1, \quad \lambda_{v,i} \geq 0.$$

For $T \subseteq [m_v]$, write $\lambda_v(T) := \sum_{i \in T} \lambda_{v,i}$. If an edge pairs port h of v with port h' of w , impose

$$\lambda_v(S_{v,h}) + \lambda_w(S_{w,h'}) = 1. \tag{1}$$

Let $P(I)$ be the resulting polytope.

Every Eulerian orientation $\sigma \in \text{EO}(G)$ with nonzero weight—that is, with $f_v(\sigma|_{E(v)}) \neq 0$ for every vertex v —induces an integral feasible point of $P(I)$ by putting all mass, at each vertex v , on the local support string $\sigma|_{E(v)}$. Hence, if $P(I) = \emptyset$, then no Eulerian orientation has nonzero weight, and the partition function of I is zero.

Definition 3.1. *Assume $P(I) \neq \emptyset$. For a vertex v , define*

$$\Gamma_v := \{i \in [m_v] : \max\{\lambda_{v,i} : \lambda \in P(I)\} > 0\},$$

and define the corresponding LP candidate set

$$U_v := \{\alpha_{v,i} : i \in \Gamma_v\} \subseteq \text{supp}(f_v).$$

Lemma 3.2 (Computing LP candidates). *Let \mathcal{F} be fixed and finite. Given an instance I of $\#\text{EO}(\mathcal{F})$ with $P(I) \neq \emptyset$, all candidate sets U_v can be computed in time polynomial in $|I|$.*

Proof. Let $m_{\max} = \max_{f \in \mathcal{F}} |\text{supp}(f)|$ and $d_{\max} = \max_{f \in \mathcal{F}} \text{arity}(f)$; both are constants. The LP $P(I)$ has at most $m_{\max}|V(G)|$ variables, one normalization constraint per vertex, and one edge constraint per edge, plus nonnegativity bounds. Since all arities are bounded by d_{\max} , the number of edges is $O(|V(G)|)$. All coefficients and right-hand sides lie in $\{-1, 0, 1\}$ because the LP depends only on the supports of the signatures, not on their complex values.

For every pair (v, i) , we solve the rational linear program $\max\{\lambda_{v,i} : \lambda \in P(I)\}$. Linear programming over rationals runs in time polynomial in the bit complexity of the input, and the optimum can be compared with zero exactly. We include $\alpha_{v,i}$ in U_v if and only if this optimum is strictly positive. There are at most $m_{\max}|V(G)| = O(|I|)$ such LPs, so the total running time is polynomial. \square

3.1 The algorithm

We first state the algorithm. The proof that the candidate sets U_v are affine is postponed: for the special signature f_{56} it is proved in Section 4, and in full generality it is proved in Section 5.

Algorithm. Given an instance I over a fixed finite signature set \mathcal{F} satisfying the tractability conditions of Theorem 1.1:

COUNTINGEO(I)	
1	if $P(I) = \emptyset$ then return 0 ▷ feasibility check
2	for each vertex $v \in V(G)$ do
3	compute the local candidate set U_v from $P(I)$ ▷ Lemma 3.2
4	find a pairing P_v with $U_v \subseteq \text{EOP}(P_v)$ ▷ constant enumeration
5	choose an orientation \vec{P}_v of P_v
6	set $g_v \leftarrow (f_v)_{\vec{P}_v}$ ▷ induced signature
7	construct the Boolean #CSP instance J over ▷ reduction
	$\{g_v : v \in V(G)\} \cup \{=_2, \neq_2\}$
8	return the value of J ▷ tractable evaluation

Since \mathcal{F} is fixed, the arity of every f_v is bounded by a constant. Hence the number of pairings of $\text{Var}(f_v)$ is also a constant depending only on \mathcal{F} , and a compatible pairing P_v can be found by brute force in polynomial time.

The pairing P_v exists by Lemma 2.15. Indeed, once U_v is proved to be affine, the restricted signature $f_v|_{U_v}$ is support-affine, and hence pairwise opposite. Therefore there exists a pairing P_v such that

$$U_v = \text{supp}(f_v|_{U_v}) \subseteq \text{EOP}(P_v).$$

After choosing an orientation \vec{P}_v of each P_v , every original edge becomes a binary equality or disequality constraint, namely $=_2$ or \neq_2 , between the corresponding pairing variables.

The next proposition records the correctness of the algorithm once the affine-candidate property is available.

Proposition 3.3 (Correctness assuming affine candidates). *Suppose that, whenever $P(I) \neq \emptyset$, the LP candidate set U_v is a nonempty affine subspace contained in $\text{supp}(f_v)$ for every vertex v . Then the algorithm above computes $Z(I)$ in polynomial time, provided $\mathcal{F} \subseteq \text{EO}^{\text{af}}$ or $\mathcal{F} \subseteq \text{EO}^{\mathcal{P}}$.*

Proof. Let $S_v(I)$ be the set of local strings appearing at v in nonzero global configurations of I . If $\alpha \in S_v(I)$, then there is a nonzero global configuration σ of I such that $\sigma|_{E(v)} = \alpha$. This configuration induces an integral feasible point of $P(I)$ by putting unit mass, at each vertex u , on the local string $\sigma|_{E(u)}$. In particular, this feasible point puts unit mass on α at v . Hence the maximum LP mass of α at v is positive, and therefore $S_v(I) \subseteq U_v$.

Since U_v is affine and $U_v \subseteq \text{supp}(f_v) \subseteq \text{HW}^=$, the restricted signature $f_v|_{U_v}$ is support-affine. By Lemma 2.15, there exists a pairing P_v of the ports of v such that $U_v \subseteq \text{EOP}(P_v)$. Choose an orientation

$$\vec{P}_v = ((p_1^0, p_1^1), \dots, (p_{d_v}^0, p_{d_v}^1))$$

of this pairing, and define the induced Boolean signature $g_v = (f_v)_{\vec{P}_v}$.

We now describe the Boolean #CSP instance J . For every vertex v and every oriented pair (p_j^0, p_j^1) of \vec{P}_v , introduce one Boolean variable $y_{v,j}$. An assignment $y_v = (y_{v,1}, \dots, y_{v,d_v})$ determines a full assignment $\tau_v(y_v)$ to all ports of v by

$$\tau_v(y_v)(p_j^0) = y_{v,j}, \quad \tau_v(y_v)(p_j^1) = 1 - y_{v,j}.$$

The local signature placed at v in J is $g_v(y_v) = f_v(\tau_v(y_v))$.

It remains to encode the original edge constraints. Let e be an edge joining port h of v and port h' of w . Suppose that h belongs to the j -th pair of \vec{P}_v , and h' belongs to the k -th pair of \vec{P}_w . Define

$$\epsilon_{v,h} = \begin{cases} 0, & h = p_j^0, \\ 1, & h = p_j^1, \end{cases} \quad \epsilon_{w,h'} = \begin{cases} 0, & h' = p_k^0, \\ 1, & h' = p_k^1. \end{cases}$$

Then the values assigned to the two ports are

$$\tau_v(y_v)(h) = y_{v,j} \oplus \epsilon_{v,h}, \quad \tau_w(y_w)(h') = y_{w,k} \oplus \epsilon_{w,h'}.$$

In the original $\#EO$ instance, the two endpoint values of every edge must be complementary. Therefore the edge e imposes $\tau_v(y_v)(h) \oplus \tau_w(y_w)(h') = 1$. Equivalently,

$$y_{v,j} \oplus y_{w,k} = 1 \oplus \epsilon_{v,h} \oplus \epsilon_{w,h'}.$$

If the right-hand side is 0, we put an equality constraint $=_2$ between $y_{v,j}$ and $y_{w,k}$; if the right-hand side is 1, we put a disequality constraint \neq_2 between them. Thus every original edge becomes either an equality or a disequality constraint between the retained pair variables.

We claim that this construction preserves the partition function. First take an assignment of the Boolean $\#CSP$ instance J with nonzero weight. For every vertex v , the local factor $g_v(y_v)$ is nonzero, so $\tau_v(y_v) \in \text{supp}(f_v)$. Moreover, the binary equality and disequality constraints ensure that, for every original edge, the two endpoint port values are complementary. Hence the assignments $\tau_v(y_v)$ over all vertices reconstruct a global Eulerian configuration of the original instance I . Since all local factors are nonzero, this is a nonzero global configuration. Therefore its local string at v belongs to $S_v(I) \subseteq U_v$.

Conversely, let σ be a nonzero global configuration of I . Then for every vertex v ,

$$\sigma|_{E(v)} \in S_v(I) \subseteq U_v \subseteq \text{EOP}(P_v).$$

Thus the two ports in every pair of P_v receive opposite values under $\sigma|_{E(v)}$. Hence $\sigma|_{E(v)}$ uniquely determines an assignment to the retained variables by $y_{v,j} := \sigma(p_j^0)$. Since σ satisfies edge complementarity in the original $\#EO$ instance, the corresponding equality or disequality constraint in J is satisfied for every edge. Therefore σ gives a unique assignment of the Boolean $\#CSP$ instance J .

The two constructions above are inverse to each other on nonzero configurations. Moreover, they preserve weights: for each corresponding pair of configurations and for every vertex v ,

$$g_v(y_v) = f_v(\tau_v(y_v)) = f_v(\sigma|_{E(v)}).$$

Thus the product of local weights is preserved, and the partition function of J is equal to $Z(I)$.

Finally, if $\mathcal{F} \subseteq \text{EO}^{\mathcal{A}}$, then for every oriented pairing \vec{P}_v the induced signature $g_v = (f_v)_{\vec{P}_v}$ belongs to \mathcal{A} . If $\mathcal{F} \subseteq \text{EO}^{\mathcal{P}}$, then every g_v belongs to \mathcal{P} . Since $=_2, \neq_2 \in \mathcal{A} \cap \mathcal{P}$, the Boolean $\#CSP$ instance J is over a tractable signature class. Hence J can be evaluated in polynomial time, and therefore the algorithm computes $Z(I)$ in polynomial time. \square

4 Warm-up: the signature f_{56}

We now prove the affine-candidate property for the motivating example f_{56} , illustrating the trilinear product argument in the smallest useful case.

Sign representation for the proofs. Throughout §4–§5, we work in the sign domain $\{\pm 1\}$. For a bit string $\alpha \in \{0, 1\}^{2d}$, its *sign encoding* is $\sigma = 2\alpha - 1 \in \{\pm 1\}^{2d}$ (coordinatewise), and we write $\Delta(\sigma) := \sum_h \sigma_h = 2 \text{wt}(\alpha) - 2d$ for the *imbalance*. Under this encoding:

- (i) Ternary XOR becomes coordinatewise product: $\alpha \oplus \beta \oplus \gamma \leftrightarrow \sigma \cdot \tau \cdot \nu$.
- (ii) The EO condition $\text{wt}(\alpha) = d$ becomes $\Delta(\sigma) = 0$; the sets $\text{HW}^>$ and $\text{HW}^<$ correspond to $\Delta(\sigma) > 0$ and $\Delta(\sigma) < 0$.
- (iii) Edge complementarity $\alpha_v(h) \neq \alpha_w(h')$ becomes $\sigma_v(h) + \sigma_w(h') = 0$.

The LP in sign form. Given a feasible LP point $\lambda \in P(I)$, define the *sign marginal* at vertex v , port h :

$$m_{v,h}(\lambda) := \sum_{i=1}^{m_v} \lambda_{v,i} \sigma_{v,i}(h) = 2\lambda_v(S_{v,h}) - 1.$$

The LP edge constraint (1) translates to

$$m_{v,h} + m_{w,h'} = 0 \tag{2}$$

whenever port h of v is paired with port h' of w .

Following [MWX25b], define the matrices

$$H_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The support matrix of f_{56} is $R_{56} := [H_0 \ H_2^{\rightarrow 4} \ H_4^{\rightarrow 3}]$, where $A^{\rightarrow k}$ denotes the concatenation of k copies of A . Let $\text{supp}(f_{56}) = \{a_1, a_2, a_3, a_4, a_5\}$ be the five rows of R_{56} . Each row has Hamming weight 28.

Lemma 4.1 (The cubic calculation for f_{56}). *Let $\lambda = (\lambda_1, \dots, \lambda_5)$ be a distribution on $\text{supp}(f_{56})$. For each port h , the sign marginal is*

$$m_h := \sum_{i=1}^5 \lambda_i \sigma_i(h),$$

where $\sigma_i = 2a_i - 1 \in \{\pm 1\}^{56}$. Define

$$\Psi_{56}(\lambda) := \frac{1}{8} \sum_{h=1}^{56} m_h^3.$$

Then

$$\Psi_{56}(\lambda) = 3 \sum_{1 \leq i < j < k \leq 5} \lambda_i \lambda_j \lambda_k \geq 0.$$

In particular, $\Psi_{56}(\lambda) = 0$ if and only if at most two entries of λ are positive.

Proof. Expanding the cube,

$$\Psi_{56}(\lambda) = \frac{1}{8} \sum_{\mathbf{i} \in [5]^3} \left(\prod_{t=1}^3 \lambda_{i_t} \right) \sum_{h=1}^{56} \prod_{t=1}^3 \sigma_{i_t}(h).$$

For a triple $\mathbf{i} = (i_1, i_2, i_3)$, the inner sum depends only on the odd-occurrence set $\text{Odd}(\mathbf{i})$. If $\text{Odd}(\mathbf{i}) = \{i\}$, the coordinatewise product reduces to σ_i itself, so the inner sum is $\sum_h \sigma_i(h) = \Delta(\sigma_i) = 0$ by (ii).

Now let $T \subseteq [5]$ have size 3. In one copy of H_2 , the xor of the rows indexed by T has weight $|T|(5 - |T|) = 6$, because the columns of H_2 are the two-subsets of $[5]$. In one copy of H_4 , the same xor has weight $5 - |T| = 2$. Hence $\text{wt}(\bigoplus_{i \in T} a_i) = 4 \cdot 6 + 3 \cdot 2 = 30$, so $\sum_h \prod_{i \in T} \sigma_i(h) = 2 \cdot 30 - 56 = 4$. For each fixed three-set T , there are $3!$ ordered triples with odd-occurrence set T , giving coefficient $\frac{1}{8} \cdot 3! \cdot 4 = 3$. This gives the displayed identity. \square

Proposition 4.2 (Affine candidates for f_{56}). *For every instance of $\#EO(\{f_{56}\})$ with $P(I) \neq \emptyset$, every LP candidate set U_v has size at most two. In particular, every U_v is an affine subspace of \mathbb{F}_2^{56} contained in $\text{supp}(f_{56})$.*

Proof. Let $\lambda \in P(I)$. By (2), $m_{v,h} + m_{w,h'} = 0$ on every edge-port pair. Therefore the cubic terms cancel edge by edge:

$$\sum_v \Psi_{56}(\lambda_v) = \frac{1}{8} \sum_v \sum_{h=1}^{56} m_{v,h}^3 = 0,$$

since each edge contributes $m^3 + (-m)^3 = 0$. By Lemma 4.1, every local term is nonnegative, so $\Psi_{56}(\lambda_v) = 0$ for every vertex v and every $\lambda \in P(I)$.

Suppose, for contradiction, that U_v contains three distinct rows a_i, a_j, a_k . By the definition of U_v , there exist feasible LP points $\lambda^{(i)}, \lambda^{(j)}, \lambda^{(k)} \in P(I)$ with positive mass on a_i, a_j, a_k at v , respectively. Their average $\lambda^* := \frac{1}{3}(\lambda^{(i)} + \lambda^{(j)} + \lambda^{(k)})$ is feasible and has positive i, j, k coordinates at v . Then Lemma 4.1 gives $\Psi_{56}(\lambda^*) > 0$, contradicting the preceding paragraph. Thus $|U_v| \leq 2$. A nonempty subset of \mathbb{F}_2^n of size at most two is affine, and by definition $U_v \subseteq \text{supp}(f_{56})$. \square

Corollary 4.3 (Correctness on f_{56}). *Since f_{56} admits an up XOR₃-quasi-polymorphism and belongs to $EO^{\mathcal{P}}$ (as established in [MWX25b]), the algorithm of Section 3.1 computes $\#EO(\{f_{56}\})$ in polynomial time.*

Proof. The affine-candidate hypothesis in Proposition 3.3 follows from Proposition 4.2. Since $f_{56} \in EO^{\mathcal{P}}$, every paired restriction selected by the algorithm is product-type, so the final Boolean $\#CSP$ instance is tractable. \square

5 The general XOR polymorphism argument

The preceding f_{56} proof used the scalar cubic potential. The general proof keeps the same “degree three” idea but uses three possibly different LP feasible points and pushes them forward by coordinatewise product in the sign domain.

Lemma 5.1 (Product pushforward). *Assume all signatures in \mathcal{F} admit a common XOR₃-quasi-polymorphism (all up or all down). Let $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)} \in P(I)$. For each vertex v , define a distribution μ_v on $\{\pm 1\}^{2d_v}$ by*

$$\mu_v(\rho) := \sum_{\substack{i_1, i_2, i_3 \in [m_v] \\ \sigma_{v, i_1} \cdot \sigma_{v, i_2} \cdot \sigma_{v, i_3} = \rho}} \lambda_{v, i_1}^{(1)} \lambda_{v, i_2}^{(2)} \lambda_{v, i_3}^{(3)},$$

where $\sigma_{v, i} = 2\alpha_{v, i} - 1$ and the product is coordinatewise. Then every μ_v is supported on $\text{supp}(f_v)$ (identified with its sign encoding), and the induced vector μ is a feasible point of $P(I)$.

Proof. Each μ_v is a probability distribution because it is the pushforward of the product distribution $\lambda_v^{(1)} \otimes \lambda_v^{(2)} \otimes \lambda_v^{(3)}$ under the coordinatewise product map. After support containment is proved, we identify μ_v with a vector indexed by $\text{supp}(f_v)$.

For $t \in \{1, 2, 3\}$, the sign marginal at vertex v , port h is

$$m_{v, h}^{(t)} := \sum_{i=1}^{m_v} \lambda_{v, i}^{(t)} \sigma_{v, i}(h).$$

By (2), $m_{v, h}^{(t)} + m_{w, h'}^{(t)} = 0$ whenever port h of v is paired with port h' of w .

Under μ_v , the sign marginal at port h is

$$\sum_{\rho} \mu_v(\rho) \rho_h = m_{v, h}^{(1)} \cdot m_{v, h}^{(2)} \cdot m_{v, h}^{(3)},$$

because the h -th coordinate of a coordinatewise product is the product of the h -th coordinates. Therefore, for every edge-port pair $((v, h), (w, h'))$,

$$m_{w, h'}^{(1)} m_{w, h'}^{(2)} m_{w, h'}^{(3)} = (-m_{v, h}^{(1)}) (-m_{v, h}^{(2)}) (-m_{v, h}^{(3)}) = -m_{v, h}^{(1)} m_{v, h}^{(2)} m_{v, h}^{(3)}.$$

Thus the pushforward satisfies all LP edge constraints (2), once it is known to be supported on the local supports.

It remains to prove support containment. We treat the case where every f_v admits an up quasi-polymorphism; the down case is symmetric. Every string in the support of μ_v is the coordinatewise product of three sign strings in $\text{supp}(f_v)$, which by (i) corresponds to the ternary XOR of the Boolean originals. Since f_v admits an up XOR₃-quasi-polymorphism, such a product has nonnegative imbalance: $\mu_v(\rho) > 0 \Rightarrow \Delta(\rho) \geq 0$. Therefore $\mathbb{E}_{\rho \sim \mu_v} [\Delta(\rho)] \geq 0$ for every v . On the other hand,

$$\begin{aligned} \sum_v \mathbb{E}_{\rho \sim \mu_v} [\Delta(\rho)] &= \sum_v \sum_{h=1}^{2d_v} m_{v, h}^{(1)} m_{v, h}^{(2)} m_{v, h}^{(3)} \\ &= 0, \end{aligned}$$

because for an edge joining ports (v, h) and (w, h') , writing $a_t = m_{v, h}^{(t)}$, we have $m_{w, h'}^{(t)} = -a_t$, so the two endpoint contributions sum to $a_1 a_2 a_3 + (-a_1)(-a_2)(-a_3) = 0$. Hence each local expectation is zero. Since every appearing imbalance is nonnegative, μ_v can put positive mass only on zero-imbalance strings, i.e. strings with $\Delta(\rho) = 0$. By the sign-domain equivalence above, zero imbalance is exactly the EO condition. A zero-imbalance string in $\text{supp}(f_v) \cup \text{HW}^>$ must lie in $\text{supp}(f_v)$. Therefore μ_v is supported on $\text{supp}(f_v)$.

For the down quasi-polymorphism case, all appearing imbalances are nonpositive, their total sum is again zero, and the same argument forces $\Delta(\rho) = 0$ for every ρ in the support. \square

Lemma 5.2 (XOR polymorphism of LP candidates). *Assume all signatures in \mathcal{F} admit a common XOR₃-quasi-polymorphism (all up or all down), and let $P(I) \neq \emptyset$. Then, for every vertex v , the LP candidate set U_v is nonempty and satisfies*

$$\alpha, \beta, \gamma \in U_v \implies \alpha \oplus \beta \oplus \gamma \in U_v.$$

Consequently, U_v is an affine subspace of $\mathbb{F}_2^{2d_v}$ contained in $\text{supp}(f_v)$.

Proof. Nonemptiness follows from $P(I) \neq \emptyset$ and $\sum_i \lambda_{v,i} = 1$. Take $\alpha_{v,a}, \alpha_{v,b}, \alpha_{v,c} \in U_v$. By definition of U_v , there exist feasible points $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)} \in P(I)$ with $\lambda_{v,a}^{(1)} > 0$, $\lambda_{v,b}^{(2)} > 0$, and $\lambda_{v,c}^{(3)} > 0$. Apply Lemma 5.1. The resulting feasible point μ assigns positive mass at v to $\delta := \alpha_{v,a} \oplus \alpha_{v,b} \oplus \alpha_{v,c}$. Indeed, the triple (a, b, c) contributes a positive amount to $\mu_v(\delta)$. Since μ_v is supported on $\text{supp}(f_v)$ and δ has positive mass in a feasible LP point, $\delta \in U_v$. □

Theorem 5.3 (Tractability). *Let \mathcal{F} be a finite set of EO signatures. Suppose that*

- (i) \mathcal{F} admits a common XOR₃-quasi-polymorphism;
- (ii) $\mathcal{F} \subseteq \text{EO}^{\mathcal{A}}$ or $\mathcal{F} \subseteq \text{EO}^{\mathcal{P}}$.

Then $\#\text{EO}(\mathcal{F}) \in \text{FP}$.

Proof. Let I be an instance of $\#\text{EO}(\mathcal{F})$. If $P(I) = \emptyset$, the algorithm outputs 0, which is correct because any globally nonzero labeling would yield an integral feasible LP point. If $P(I) \neq \emptyset$, then Lemma 5.2 gives that every U_v is a nonempty affine subspace contained in $\text{supp}(f_v)$. Therefore Proposition 3.3 applies and the algorithm computes $Z(I)$ in polynomial time. □

Proof of Theorem 1.1. If the two stated conditions hold, then tractability follows from Theorem 5.3. If at least one condition fails, then $\#\text{P}$ -hardness follows from Theorem 2.13. □

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