Strong Spatial Mixing for Continuous Spin Systems with Multi-Interactions

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Abstract

We study spatial correlation decay, focusing on strong spatial mixing (SSM), for continuous spin systems with multi-interactions on hypergraphs. For several classes of discrete spin systems, zero-freeness of the partition function and SSM have been related in both directions. For continuous spin systems, this relationship is less understood. In this paper, we prove that a uniform pinned zero-free disk about $\beta = 0$ implies SSM throughout that disk for continuous spin systems with bounded interactions. Combined with the zero-freeness result of Barvinok [4], this gives SSM for systems whose interactions are 1-Lipschitz in the Hamming metric, on hypergraphs with maximum degree Δ and maximum edge size r, for all complex β with $|\beta| < 1/(3\Delta\sqrt{r-1})$. The SSM constants are uniform over the system size, the product reference measures, and all valid product boundary events.

A key technical contribution of this paper is a divisibility relation for partition functions of continuous spin systems. We follow the local dependence of coefficients (LDC) framework developed for discrete systems and extend its divisibility step to integral partition functions. The Taylor coefficients in our divisibility identity are sums of integrals of products of interactions, with repeated hyperedges allowed. A measure-preserving swap of two configurations, together with a reindexing of the interaction factors, cancels all contributions below the distance between the boundary supports. This shows how the LDC and zero-freeness approaches fit together; algorithmic running times additionally depend on how the continuous interactions and the required integrals are represented and computed.

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1 Introduction

Spin systems arise in statistical physics to model interactions on hypergraphs. Let $\mathcal{H} = (V, E)$ be a finite hypergraph, with $V = [n]$. For each $i \in V$, let $(X_i, \mathcal{B}_i, \nu_i)$ be a measurable probability space, and set $\Omega := \prod_{i \in V} X_i$ and $\nu := \bigotimes_{i \in V} \nu_i$. For each edge $e \in E$, set $X_e := \prod_{i \in e} X_i$, write $x_e := (x_i)_{i \in e}$, and let $\Phi_e : X_e \to \mathbb{C}$ be a bounded measurable interaction. For inverse temperature $\beta \in \mathbb{C}$, define the Hamiltonian and Boltzmann weight by

$$H(x) := \sum_{e \in E} \Phi_e(x_e), \qquad \omega_\beta(x) := \exp\bigl(-\beta H(x)\bigr).$$

The partition function is

$$Z_{\mathcal{H}}(\beta) := \int_\Omega \omega_\beta(x)\, d\nu(x).$$

When $\beta \geq 0$ and every Φ_e is real-valued, the weights are nonnegative and $Z_{\mathcal{H}}$ is the normalizing constant of the corresponding Gibbs distribution.

When Ω is finite, this integral is a finite sum of exponentials in β and, in many standard formulations, can be expressed as a polynomial in suitable activity variables. For example, if each $X_i = \{0, 1\}$ with the uniform measure, one obtains a two-state spin model[1]. For continuous state spaces, the partition function remains an integral and does not naturally admit a polynomial structure.

Correlation decay is a classical tool for studying phase transitions in spin systems. It refers to influences or correlations decaying exponentially with separation. More formally, for an infinite graph such as an infinite regular tree, one can identify regions in the parameter space where, under the Gibbs distribution, correlations decay exponentially as a function of the distance between them. Another standard notion is analyticity of a local branch of the free energy $\log Z_{\mathcal{H}}$ on zero-free domains, which is closely related to the study of zeros of the partition function. One of the earliest and best-known results regarding zero-freeness is the Lee–Yang theorem [34, 12] for the Ising model, a special case of the 2-spin system. This result was later extended to more general models [2, 26, 30, 20, 15].

For discrete spin systems, zero-freeness and strong spatial mixing (SSM) can underpin deterministic approximation schemes under additional model and representation assumptions. SSM requires the effect of changing a boundary condition on a local marginal to decay exponentially with the distance to the sites where the two boundary conditions differ, uniformly over their common constraints. Examples include Weitz's algorithm [33] and Barvinok-type interpolation methods [3]. Motivated by these algorithmic implications, substantial recent work has established SSM [36, 13, 14, 31] and zero-freeness [23, 5, 18, 17, 16, 24, 8, 22, 10, 28] results for discrete spin systems.

Moreover, inherent connections between zero-freeness and correlation decay have been established for discrete spin systems. In seminal works [6, 7], Dobrushin and Shlosman related appropriate forms of correlation decay and complete analyticity to zero-freeness for lattice models. Their lattice argument uses the polynomial growth of lattice neighborhoods, a feature that does not directly extend to general bounded-degree or non-amenable graph families. Only recently have further connections been established beyond lattice models. In [28], correlation decay at real parameters is extended to complex neighborhoods where the partition function remains zero-free. In the opposite direction, Gamarnik [9] showed that zero-freeness implies a weak form of SSM for the hard-core model and certain graph homomorphism models under structural assumptions. Regts [25] strengthened this implication to full SSM. This line of work was further extended to general 2-spin systems with complex parameters [29, 27]. Under model-specific hypotheses, in particular suitable pinned zero-freeness statements, these works derive SSM on the corresponding parameter ranges.

[1] Typically, a two-state spin system is defined on graphs with edges representing binary interactions. In this paper, we also allow hypergraphs with multi-spin interactions.

For continuous spin systems, results on point-to-point correlation decay (which is weaker than SSM) [7, 19, 35, 32, 1] and zero-freeness results [15, 20, 2, 26] have been established, mostly for special structures such as lattice models. There is a recent zero-freeness result [4] for discrete and continuous spin systems on general hypergraphs with bounded vertex degree and bounded hyper-edge size, under the stated Lipschitz and product-measure hypotheses. Specifically, it proves that $Z_{\mathcal{H}}(\beta) \neq 0$ for any complex β with $|\beta| < \frac{1}{3\,\Delta\,\sqrt{r-1}}$, where Δ is the maximum vertex degree of $\mathcal{H}$ and r is the maximum hyper-edge size. The algorithmic implications in the continuous setting require additional computational assumptions. Barvinok [4, p. 750 and Section 5] reduces approximation inside a fixed smaller zero-free disk to computing low-order interaction moments, yielding quasi-polynomial algorithms when those moments can be evaluated efficiently. In suitable settings with bounded degree and bounded interaction arity, including explicitly represented finite-spin instances, the Patel–Regts method [21] improves this to polynomial time. Thus zero-freeness does have algorithmic consequences for continuous systems, although it does not by itself provide an FPTAS for arbitrary measurable interactions and reference measures. To the best of our knowledge, the cited literature does not establish a pinned zero-freeness-to-SSM implication for general bounded-degree hypergraphs with continuous spins.

In this paper, we establish a direct zero-freeness-to-SSM implication for continuous spin systems. Under the Lipschitz conditions in Barvinok's zero-freeness result [4], we prove SSM throughout the corresponding pinned zero-free disk for hypergraphs with bounded vertex degree and bounded hyperedge size.

▶ **Theorem 1.1** (Main). *Fix integers $\Delta \geq 1$ and $r \geq 2$. For every complex β with $|\beta| < 1/(3\Delta\sqrt{r-1})$, the family of spin systems on all finite hypergraphs of maximum degree at most Δ and maximum edge size at most r, with product reference probability measures and each interaction Φ_e 1-Lipschitz in Hamming distance as in Theorem 2.5, satisfies SSM at β in the sense of Definition 2.2. The constants $C > 0$ and $q > 1$ depend only on β, Δ, r, and are uniform over the hypergraph, reference measures, interaction functions, vertex, and valid boundary events.*

Classical high-temperature results for continuous spin systems include Israel's work [11] on analyticity for lattice systems and, under additional structural assumptions, exponential clustering. These conclusions involve different hypotheses and mixing notions. Theorem 1.1 gives SSM uniformly over all valid product boundary events for finite hypergraphs under the stated Hamming-Lipschitz condition.

A key step in deriving SSM from zero-freeness is to establish a property called *local dependence of coefficients* (LDC). This property states that the low-order Taylor coefficients of a conditional marginal are unchanged when the boundary conditions agree sufficiently close to the observed vertex. Our proof follows the framework of Regts [25]: establish local dependence of Taylor coefficients, use zero-freeness to bound conditional marginals uniformly on compact subsets, and apply a complex-analytic truncation estimate to obtain SSM. We retain this analytic framework, including the use of Montel's theorem. The step requiring an extension is the local dependence statement for integral partition functions under arbitrary valid product boundary events. Our proof of this step builds on the combinatorial divisibility approach of Shao and Shi [27].

Taylor expansion alone does not establish the required locality. In our divisibility identity, each Taylor coefficient of the partition-function combination is a sum of integrals of products of interactions indexed by tuples of hyperedges, which may contain repetitions. For a term of order k smaller than the distance between two boundary supports, the selected hyperedges cannot connect those supports. We swap the two configurations on the components meeting one support and simultaneously exchange the corresponding factor labels. The product reference measure is preserved, the two constrained integration domains are exchanged, and the coefficient contributions cancel. This proves the divisibility relation in Lemma 3.2 without discretizing the spin spaces. Conditioning on a common product event then gives LDC, while pinned zero-freeness supplies a bound uniform over all measurable one-site events and valid boundary events.

For approximate counting, the significance of this result is that a zero-free disk also certifies uniform insensitivity of local marginals to distant product boundary constraints. For fixed β, Δ, r, Theorem 1.1 bounds the effect of replacing constraints at distance at least L by Cq^{-L}; hence $L = O(\log(1/\varepsilon))$ suffices for additive marginal error ε. This is the error-control ingredient used in correlation-decay methods. Turning it into an algorithm still requires an efficient way to evaluate the resulting continuous integrals and control numerical precision. Likewise, Taylor interpolation requires efficient access to coefficient integrals, a requirement distinct from proving that those coefficients are local. Our contribution is this structural implication; we do not derive a new approximation scheme or running-time bound here.

2 Preliminaries

2.1 Partition function for continuous models

We first give a detailed formal definition of the partition function of continuous spin systems. The notation may differ slightly from that in the Introduction.

Let $V = [n] := \{1, \dots, n\}$ be the vertex set, and let $\mathcal{H} = (V, E)$ be a finite hypergraph on V, where $E \subseteq 2^V \setminus \{\varnothing\}$ is a collection of hyperedges. Empty hyperedges contribute only constants to the Hamiltonian and may be removed without changing the Gibbs marginals or partition-function zeros. For each $i \in V$, let $(X_i, \mathcal{B}_i, \nu_i)$ be a probability space with reference measure ν_i. Define

$$\Omega := \prod_{i \in V} X_i, \qquad \mathcal{B} := \bigotimes_{i \in V} \mathcal{B}_i, \qquad \nu := \bigotimes_{i \in V} \nu_i.$$

For each hyperedge $e \in E$, let $X_e := \prod_{i \in e} X_i$ and let $\Phi_e : X_e \to \mathbb{C}$ be a measurable interaction term. For $x = (x_i)_{i \in V} \in \Omega$, write $x_e := (x_i)_{i \in e} \in X_e$. For any $x = (x_i)_{i \in V} \in \Omega$, the Hamiltonian is $H(x) = \sum_{e \in E} \Phi_e(x_e)$. For $\beta \in \mathbb{C}$, define the partition function

$$Z_{\mathcal{H}}(\beta) := \int_{\Omega} e^{-\beta H(x)} \, d\nu(x).$$

Throughout, we assume that the interaction functions are bounded. This is automatic under the Hamming-Lipschitz condition of Theorem 2.5: for any fixed $x_e^0 \in X_e$, changing coordinates one at a time gives $|\Phi_e(x_e)| \le |\Phi_e(x_e^0)| + |e|$. Thus H is bounded for every finite instance, and the exponential series converges absolutely and uniformly on every bounded set of β values, uniformly in x. Consequently, all partition functions, including those restricted to measurable events, are entire in β and may be expanded by integrating the Taylor series term by term. When clear from context, we write $Z(\beta)$ for $Z_{\mathcal{H}}(\beta)$.

Fix a subset of vertices (the pinned set) $\Lambda \subseteq V$. For each $i \in \Lambda$, choose a measurable set $A_i \in \mathcal{B}_i$. Define the associated *cylinder set* (or pinned event):

$$A_\Lambda := \{x \in \Omega : \ x_i \in A_i \text{ for all } i \in \Lambda\}.$$

In product form,

$$A_\Lambda = \Big(\prod_{i \in \Lambda} A_i\Big) \times \Big(\prod_{i \notin \Lambda} X_i\Big).$$

We call A_Λ *valid* if $\nu(A_\Lambda) > 0$. When no confusion arises, we write A for A_Λ. For a valid cylinder set A and $\beta \in \mathbb{C}$, define the conditional partition function

$$Z_{\mathcal{H},A}(\beta) := \int_\Omega e^{-\beta H(x)} \mathbf{1}_A(x)\, d\nu(x) = \int_A e^{-\beta H(x)}\, d\nu(x),$$

where $\mathbf{1}_A$ is the indicator of A, i.e.,

$$\mathbf{1}_A(x) = \prod_{i\in\Lambda} \mathbf{1}_{A_i}(x_i) = \begin{cases} 1 & \text{if } x_i \in A_i \text{ for all } i \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

▶ Remark 2.1. The condition $\nu(A) > 0$ excludes degenerate cases in which $Z_{\mathcal{H},A}(\beta) \equiv 0$.

Given integers $\Delta \geq 1$ and $r \geq 2$, define $\mathbb{H}(\Delta, r)$ as the family of finite hypergraphs with maximum degree at most Δ and maximum edge size at most r, and define $\mathbb{D}(\Delta, r) := \left\{\beta \in \mathbb{C} :\ |\beta| < \frac{1}{3\Delta\sqrt{r-1}}\right\}$.

In discrete models, each X_i is a finite set, and the partition function reduces to a finite sum over configurations. We allow arbitrary measurable spin spaces; intervals are a motivating continuous example. When no confusion arises, we write $Z_A(\beta)$ for $Z_{\mathcal{H},A}(\beta)$.

2.2 Conditional marginals and strong spatial mixing

For a family of discrete models on graphs $G = (V, E)$ with a finite spin set S, the *strong spatial mixing* (SSM) property requires constants $C > 0$ and $q > 1$, uniform over the family, with the following property. For every model, every vertex $v \in V$, every subset $\Lambda \subseteq V \setminus \{v\}$, and every pair of (partial) configurations $A, B \in S^\Lambda$ on Λ, let μ_v^A and μ_v^B denote the conditional marginal distributions at v under the conditional events A and B, respectively. Define the disagreement set $(A \neq B) := \{u \in \Lambda : A(u) \neq B(u)\}$. The constants must satisfy

$$\|\mu_v^A - \mu_v^B\|_{\mathrm{TV}} \leq C\, q^{-d_G(v, A\neq B)},$$

where $d_G(v, A \neq B) := \min_{u\in(A\neq B)} d_G(v, u)$ (with the convention $d_G(v, \varnothing) = +\infty$) and $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation distance.

Such a definition can be extended to continuous-spin models in a natural way.

In continuous models defined on a hypergraph $\mathcal{H} = (V, E)$, two vertices $u, v \in V$ are *adjacent* if they belong to a common hyperedge. The *hypergraph distance* $d_{\mathcal{H}}(u, v)$ is the length of a shortest incidence chain $u = v_0, e_1, v_1, \ldots, e_m, v_m = v$, where $v_{j-1}, v_j \in e_j$; it is 0 when $u = v$ and $+\infty$ when no such chain exists. Equivalently, it is the shortest-path distance in the graph whose edges connect pairs of vertices sharing a hyperedge. For vertex sets $S, T \subseteq V$, define

$$d_{\mathcal{H}}(S, T) := \min_{u\in S,\, w\in T} d_{\mathcal{H}}(u, w),$$

with the convention that the minimum is $+\infty$ if either set is empty. We write $d_{\mathcal{H}}(v, S) := d_{\mathcal{H}}(\{v\}, S)$.

For a subset $\Lambda \subseteq V$, recall that a *cylinder set* on Λ is a product event $A = \prod_{i\in\Lambda} A_i \times \prod_{i\notin\Lambda} X_i$ with each $A_i \in \mathcal{B}_i$. Assume A is *valid* and $Z_A(\beta) \neq 0$. The *conditional marginal* at v given A is the finite complex measure defined by

$$\mu_v^A(\,\cdot\,, \beta) \;:=\; \frac{Z_{A\cap\{x_v\in\,\cdot\,\}}(\beta)}{Z_A(\beta)}.$$

For complex weights this is a normalized complex measure; for real weights with a positive density it is a probability measure. When no confusion arises, we write μ_v^A for $\mu_v^A(\,\cdot\,, \beta)$.

▶ **Definition 2.2** (Strong spatial mixing). *Let $\mathfrak{F}$ be a family of finite spin systems, and fix $\beta \in \mathbb{C}$ at which every valid pinned partition function of every system in $\mathfrak{F}$ is nonzero. We say that $\mathfrak{F}$ satisfies* strong spatial mixing *(SSM) at β if there exist constants $C > 0$ and $q > 1$ such that, for every system in $\mathfrak{F}$, every vertex $v \in V$, every subset $\Lambda \subseteq V \setminus \{v\}$, and every pair of valid cylinder events A, B on Λ,*

$$\|\mu_v^A - \mu_v^B\|_{\mathrm{TV}} \;\le\; C\, q^{-d_{\mathcal{H}}(v, A \neq B)}.$$

The constants may depend on β and the parameters defining the family, but not on the individual system, v, Λ, A, or B. The disagreement set is defined by $(A \neq B) := \Big\{u \in \Lambda : \nu_u\big(A_u \triangle B_u\big) > 0\Big\}$ and we use the event-supremum convention

$$\|\mu_v^A - \mu_v^B\|_{\mathrm{TV}} := \sup_{D \in \mathcal{B}_v} |\mu_v^A(D; \beta) - \mu_v^B(D; \beta)|.$$

For probability measures this is the usual total variation distance; for complex measures it is equivalent, up to an absolute constant, to their total variation norm. We set $q^{-\infty} = 0$. When $d_{\mathcal{H}}(v, A \neq B) = \infty$, the boundary events agree almost everywhere on the connected component of v, and factorization over connected components gives $\mu_v^A = \mu_v^B$.

2.3 Complex analysis tools

In this paper, a *region* means a connected open subset of $\mathbb{C}$. For $\rho > 0$, let $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ be the disk of radius ρ, and denote its boundary by $\partial\mathbb{D}_\rho$. For functions analytic in a neighborhood of 0, with expansions $f(z) = \sum_{k \ge 0} a_k z^k$ and $g(z) = \sum_{k \ge 0} b_k z^k$, and an integer $m \ge 0$, we write $z^m \mid (f(z) - g(z))$ to mean that $a_i = b_i$ for all $0 \le i \le m - 1$. For $m = \infty$, this notation means that all Taylor coefficients agree, so f and g coincide in a neighborhood of 0.

The following standard estimate follows from Cauchy's integral formula.

▶ **Lemma 2.3.** *Let f and g be analytic on a region containing $\overline{\mathbb{D}}_\rho$, and assume that $|f(z)| \le M$ and $|g(z)| \le M$ for all $z \in \partial\mathbb{D}_\rho$. If $z^m \mid (f(z) - g(z))$ for an integer $m \ge 1$, then for any $z \in \mathbb{D}_\rho \setminus \{0\}$,*

$$|f(z) - g(z)| \le \frac{2M}{q^{m-1}(q-1)}, \qquad q := \frac{\rho}{|z|} > 1,$$

where $\overline{\mathbb{D}}_\rho := \{z \in \mathbb{C} : \ |z| \le \rho\} = \mathbb{D}_\rho \cup \partial\mathbb{D}_\rho$.

Proof. Cauchy's estimate gives $|a_k - b_k| \le 2M\rho^{-k}$ for every $k \ge 0$. Since the first m coefficients agree,

$$|f(z) - g(z)| \le 2M \sum_{k=m}^{\infty} (|z|/\rho)^k = \frac{2M}{q^{m-1}(q-1)}.$$

◀

To apply Lemma 2.3, we require a uniform bound for a family of functions. Regts [25] invokes Montel's theorem to obtain such bounds, which leads to the following result.

▶ **Lemma 2.4** (Montel-type boundedness). *Let $\mathcal{U} \subseteq \mathbb{C}$ be a region, and let $\mathcal{F}$ be a family of holomorphic maps $f : \mathcal{U} \to \mathbb{C}$ such that $f(\mathcal{U}) \subseteq \mathbb{C} \setminus \{0, 1\}$ for every $f \in \mathcal{F}$. If there exist*

$z_0 \in \mathcal{U}$ and $C_0 > 0$ such that $|f(z_0)| \le C_0$ for all $f \in \mathcal{F}$, then for every compact set $K \subseteq \mathcal{U}$ there exists $M_K < \infty$ with

$$\sup_{f\in\mathcal{F}} \sup_{z\in K} |f(z)| \le M_K.$$

Proof. Montel's theorem gives normality in the spherical metric, since every function omits $0, 1$, and ∞. Any sequence in $\mathcal{F}$ therefore has a subsequence converging locally uniformly in that metric to a meromorphic function or to the constant ∞. The bound at z_0 excludes the latter limit. The meromorphic limit cannot have a pole: near a pole, the reciprocals converge in the Euclidean metric to a holomorphic function with a zero, so Hurwitz's theorem forces that reciprocal limit to vanish identically, contradicting the finite value at z_0 and connectedness. Thus the limit is holomorphic, and convergence is locally uniform in the Euclidean metric. If the asserted bound on a compact set failed, a sequence with unbounded suprema on that set would contradict this convergence. ◀

2.4 Barvinok's zero-free regions

In the rest of the paper we view the partition function as a complex-analytic function of the inverse temperature $\beta \in \mathbb{C}$. A key structural property that will be used repeatedly is the existence of a domain in the complex plane where the partition function does not vanish.

Recall the conditional partition function for any valid cylinder set A,

$$Z_{\mathcal{H},A}(\beta) \;:=\; \int_A \exp\bigl(-\beta H(x)\bigr)\, d\nu(x).$$

We say that $\mathcal{D} \subseteq \mathbb{C}$ is a *pinned zero-free region* if $Z_{\mathcal{H},A}(\beta) \neq 0$ for all $\beta \in \mathcal{D}$ and all valid cylinder sets A.

The following theorem, due to Barvinok, provides an explicit zero-free disk for the unpinned partition function.

▶ **Theorem 2.5** (Barvinok [4]). *Let $\mathcal{H} = (V, E)$ be a hypergraph with $V = [n]$, and let $(\Omega, \mathcal{B}, \nu) = \prod_{v\in V}(X_v, \mathcal{B}_v, \nu_v)$ be the associated product probability space. Suppose the Hamiltonian decomposes as $H(x) = \sum_{e\in E} \Phi_e(x_e)$, where each measurable Φ_e is 1-Lipschitz with respect to Hamming distance on X_e, i.e.,*

$$|\Phi_e(x_e) - \Phi_e(y_e)| \le 1 \quad \text{whenever } x_e \text{ and } y_e \text{ differ in exactly one coordinate.}$$

Let $r \ge 2$ and $\Delta \ge 1$ be upper bounds on the maximum edge size and maximum vertex degree, respectively. Then $\mathbb{D}(\Delta, r) = \left\{\beta \in \mathbb{C} : \; |\beta| < \frac{1}{3\,\Delta\,\sqrt{r-1}}\right\}$ is a zero-free region for $Z_{\mathcal{H}}(\beta)$.

The same zero-free disk extends to arbitrary valid product pinnings.

▶ **Corollary 2.6.** *Under the setting of Theorem 2.5, $\mathbb{D}(\Delta, r)$ is a pinned zero-free region: for every valid cylinder set $A = \prod_{i\in V} A_i$ (with $A_i = X_i$ outside its support) satisfying $\nu(A) > 0$, we have $Z_{\mathcal{H},A}(\beta) \neq 0$ for all $\beta \in \mathbb{D}(\Delta, r)$.*

Proof. For each $i \in V$, define the normalized restricted measure $\nu_i^A(\cdot) := \nu_i(\cdot \cap A_i)/\nu_i(A_i)$ and let $\widetilde{\nu} := \bigotimes_{i\in V} \nu_i^A$. Since ν is a product measure and $A = \prod_i A_i$,

$$Z_{\mathcal{H},A}(\beta) = \int_A e^{-\beta H(x)}\, d\nu(x) = \nu(A) \int_\Omega e^{-\beta H(x)}\, d\widetilde{\nu}(x) = \nu(A)\, \widetilde{Z}_{\mathcal{H}}(\beta),$$

where $\widetilde{Z}_{\mathcal{H}}(\beta)$ is the unpinned partition function on the restricted product probability space $(\Omega, \mathcal{B}, \widetilde{\nu})$. The interaction decomposition, maximum degree, maximum edge size, and Lipschitz constants are unchanged, so Theorem 2.5 gives $\widetilde{Z}_{\mathcal{H}}(\beta) \neq 0$ for all $\beta \in \mathbb{D}(\Delta, r)$. Since $\nu(A) > 0$, we conclude $Z_{\mathcal{H},A}(\beta) \neq 0$. ◀

3 Local dependence of coefficients for continuous systems

In this section we formulate LDC for continuous-spin systems through the Taylor expansion at $\beta = 0$ of conditional marginals: coefficients below the boundary-disagreement distance are insensitive to changes farther away. We prove this by establishing a divisibility identity for pinned partition functions. The key step is to expand the exponential and use a measure-preserving swap map to pair terms and cancel them, extending the discrete divisibility argument to this non-polynomial setting.

▶ **Definition 3.1** (Local dependence of coefficients). *We say the model satisfies the* LDC property *if, for every vertex $v \in V$, every subset $\Lambda \subseteq V \setminus \{v\}$, every pair of valid cylinder events A, B on Λ, and every $D \in \mathcal{B}_v$, the Taylor series at $\beta = 0$ of $\mu_v^A(D;\beta) - \mu_v^B(D;\beta)$ is divisible by $\beta^{d_{\mathcal{H}}(v, A\neq B)}$. The disagreement set $(A \neq B)$ and the distance $d_{\mathcal{H}}(\cdot,\cdot)$ are defined in the same way as in Definition 2.2.*

We prove a divisibility relation similar to the one established for the discrete model in [27], which implies the LDC property.

▶ **Lemma 3.2** (Divisibility). *For any cylinder sets A supported on Λ_A and B supported on Λ_B where Λ_A and Λ_B are disjoint vertex sets, let $AB := A \cap B$ (which is again a cylinder set supported on $\Lambda_A \cup \Lambda_B$), and let $d := d_{\mathcal{H}}(\Lambda_A, \Lambda_B)$. Then*

$$\beta^d \;\Big|\; Z(\beta)\, Z_{AB}(\beta) - Z_A(\beta)\, Z_B(\beta).$$

All partition functions below use the convention $Z_S(\beta) := \int_S e^{-\beta H(x)}\, d\nu(x)$.

Proof. We write

$$\begin{aligned} &Z(\beta)\, Z_{AB}(\beta) - Z_A(\beta)\, Z_B(\beta) \\ &= \int_{\Omega\times AB} e^{-\beta(H(x)+H(y))}\, d\nu(x)\, d\nu(y) - \int_{A\times B} e^{-\beta(H(x)+H(y))}\, d\nu(x)\, d\nu(y). \end{aligned}$$

Expanding the exponential $e^{-\beta(H(x)+H(y))} = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} \big(H(x)+H(y)\big)^k$, and interchanging summation and integration, justified on $|\beta| \le R_0$ by the bound $\exp(2R_0\|H\|_\infty)$ for any $R_0 > 0$, we obtain $Z(\beta)\, Z_{AB}(\beta) - Z_A(\beta)\, Z_B(\beta) = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} \Delta_k$, where

$$\Delta_k := \int_{\Omega\times AB} \big(H(x)+H(y)\big)^k\, d\nu(x)\, d\nu(y) - \int_{A\times B} \big(H(x)+H(y)\big)^k\, d\nu(x)\, d\nu(y).$$

Therefore it suffices to show that $\Delta_k = 0$ for every $0 \le k < d$, which implies that the coefficients of $\beta^0, \beta^1, \ldots, \beta^{d-1}$ vanish and hence the difference is divisible by β^d.
Recall the interaction representation

$$H(z) = \sum_{e\in E} \Phi_e(z_e),$$

where each Φ_e depends only on the coordinates on the hyperedge e. Fix $k \ge 0$. First expand by choosing, for each of the k factors, whether we use $H(x)$ or $H(y)$:

$$\big(H(x)+H(y)\big)^k = \sum_{\sigma\in\{0,1\}^k} \prod_{i=1}^{k} H\big(z^{(\sigma_i)}\big), \qquad z^{(0)} := x, \quad z^{(1)} := y.$$

Next expand each H as a sum of Φ_e's:

$$\prod_{i=1}^{k} H\big(z^{(\sigma_i)}\big) = \prod_{i=1}^{k}\Big(\sum_{e\in E}\Phi_e\big(z_e^{(\sigma_i)}\big)\Big) = \sum_{\mathbf{e}=(e_1,\dots,e_k)\in E^k}\ \prod_{i=1}^{k}\Phi_{e_i}\big(z_{e_i}^{(\sigma_i)}\big).$$

Hence

$$\big(H(x)+H(y)\big)^k = \sum_{\sigma\in\{0,1\}^k}\ \sum_{\mathbf{e}\in E^k} f_{\sigma,\mathbf{e}}(x,y), \qquad f_{\sigma,\mathbf{e}}(x,y) := \prod_{i=1}^{k}\Phi_{e_i}\big(z_{e_i}^{(\sigma_i)}\big).$$

Thus

$$\Delta_k = \sum_{\sigma\in\{0,1\}^k}\ \sum_{\mathbf{e}\in E^k}\Big(\int_{\Omega\times AB} f_{\sigma,\mathbf{e}}\, d\nu^{\otimes 2} - \int_{A\times B} f_{\sigma,\mathbf{e}}\, d\nu^{\otimes 2}\Big).$$

It is enough to show that for each fixed $\mathbf{e}\in E^k$,

$$\sum_{\sigma\in\{0,1\}^k}\int_{\Omega\times AB} f_{\sigma,\mathbf{e}}\, d\nu^{\otimes 2} = \sum_{\sigma\in\{0,1\}^k}\int_{A\times B} f_{\sigma,\mathbf{e}}\, d\nu^{\otimes 2}. \tag{1}$$

Let $\mathcal{H}[\mathbf{e}]$ be the multihypergraph with hyperedge occurrences $e_1,\dots,e_k$ and vertex set $\bigcup_{i=1}^k e_i$; repeated entries of the tuple are allowed. Let $C_1,\dots,C_m$ be its connected components, and let $V(C_j)$ be the union of the hyperedges in C_j. If a component met both Λ_A and Λ_B, it would contain an incidence chain between the two sets using at most k hyperedge occurrences, contrary to $k<d$. Thus every component meets at most one of Λ_A and Λ_B.

Let

$$W := \bigcup\{\, V(C_j) : V(C_j)\cap\Lambda_A \neq \varnothing\,\}\cup\Lambda_A.$$

Then W is disjoint from Λ_B (because no component meets both Λ_A and Λ_B), and for each $i\in\{1,\dots,k\}$ we have either $e_i\subseteq W$ (if e_i lies in an A-component) or $e_i\cap W=\varnothing$ (if e_i lies in a non-A component).

Define the map $S:\Omega\times\Omega\to\Omega\times\Omega$ by swapping the two configurations on the vertex set W:

$$S(x,y) = (x',y'), \qquad x'_v = \begin{cases} y_v, & v\in W,\\ x_v, & v\notin W,\end{cases} \qquad y'_v = \begin{cases} x_v, & v\in W,\\ y_v, & v\notin W.\end{cases}$$

Because ν is a product measure, S is measure-preserving for $d\nu(x)\,d\nu(y)$.

Moreover, since A depends only on $\Lambda_A\subseteq W$ and B depends only on $\Lambda_B\subseteq V\backslash W$, we have:

$$(x,y)\in\Omega\times AB \quad\Longleftrightarrow\quad S(x,y)\in A\times B.$$

Indeed, $(x,y)\in\Omega\times AB$ means $y\in A$ on Λ_A and $y\in B$ on Λ_B; after swapping on $W\supseteq\Lambda_A$, the A-constraint moves to x' while the B-constraint stays on y' (since $W\cap\Lambda_B=\varnothing$).

Now define the index set of "A-component factors"

$$I_A := \{\, i\in\{1,\dots,k\} : e_i\subseteq W\,\},$$

and for $\sigma\in\{0,1\}^k$ define $\sigma^A\in\{0,1\}^k$ by flipping the bits on I_A:

$$\sigma_i^A = \begin{cases} 1-\sigma_i, & i\in I_A,\\ \sigma_i, & i\notin I_A.\end{cases}$$

This map $\sigma \mapsto \sigma^A$ is a bijection of $\{0,1\}^k$.

Crucially, for every (x, y) we have the identity

$$f_{\sigma,\mathbf{e}}\big(S(x,y)\big) = f_{\sigma^A,\mathbf{e}}(x,y). \tag{2}$$

To see this, note that if $i \in I_A$ then $e_i \subseteq W$ so the swap exchanges x_{e_i} and y_{e_i}, hence $\Phi_{e_i}\big(z_{e_i}^{(\sigma_i)}\big)$ becomes $\Phi_{e_i}\big(z_{e_i}^{(1-\sigma_i)}\big)$; if $i \notin I_A$ then $e_i \cap W = \varnothing$ and the factor is unchanged.

Using the change of variables $(x', y') = S(x, y)$, the fact that S maps $\Omega \times AB$ bijectively onto $A \times B$, and (2), we get

$$\begin{aligned}
\sum_{\sigma\in\{0,1\}^k} \int_{A\times B} f_{\sigma,\mathbf{e}}(x',y')\, d\nu(x')\, d\nu(y') &= \sum_{\sigma\in\{0,1\}^k} \int_{\Omega\times AB} f_{\sigma,\mathbf{e}}\big(S(x,y)\big)\, d\nu(x)\, d\nu(y) \\
&= \sum_{\sigma\in\{0,1\}^k} \int_{\Omega\times AB} f_{\sigma^A,\mathbf{e}}(x,y)\, d\nu(x)\, d\nu(y) \\
&= \sum_{\sigma\in\{0,1\}^k} \int_{\Omega\times AB} f_{\sigma,\mathbf{e}}(x,y)\, d\nu(x)\, d\nu(y),
\end{aligned}$$

where in the last step we reindex the sum using the bijection $\sigma \mapsto \sigma^A$. This proves (1).

Summing (1) over all $\mathbf{e} \in E^k$ yields $\Delta_k = 0$ for every $0 \le k < d$. For finite d, this is the required divisibility; for $d = \infty$, all Taylor coefficients vanish, so the entire function $Z(\beta)Z_{AB}(\beta) - Z_A(\beta)Z_B(\beta)$ is identically zero. ◀

Such a divisibility relationship implies the following LDC result.

▶ **Lemma 3.3.** *Let $v \in V$ be a vertex, and let A and B be two valid cylinder events supported on $\Lambda \subseteq V \setminus \{v\}$. If $(A \neq B) = \varnothing$, then $\mu_v^A(D) = \mu_v^B(D)$ for every $D \in \mathcal{B}_v$. Otherwise, for every $D \in \mathcal{B}_v$,*

$$\beta^{d_{\mathcal{H}}(v, A\neq B)} \;\Big|\; \big(\mu_v^A(D) - \mu_v^B(D)\big).$$

When $d_{\mathcal{H}}(v, A \neq B) = \infty$, the divisibility means equality in a neighborhood of $\beta = 0$; factorization over connected components gives equality wherever both conditional marginals are defined.

We first show one-sided LDC from the divisibility relation.

▶ **Lemma 3.4.** *Let $v \in V$, let A be a valid cylinder event supported on $\Lambda \subseteq V \setminus \{v\}$, and let $D \in \mathcal{B}_v$. Then*

$$\beta^{d_{\mathcal{H}}(v,\Lambda)} \mid \big(\mu_v(D) - \mu_v^A(D)\big).$$

Proof. View D as a cylinder event supported on $\{v\}$. Then

$$\begin{aligned}
\mu_v(D) - \mu_v^A(D) &= \frac{Z_D}{Z} - \frac{Z_{AD}}{Z_A} \\
&= \frac{Z_D Z_A - Z\, Z_{AD}}{Z\, Z_A}.
\end{aligned}$$

By Lemma 3.2, we have

$$\beta^{d_{\mathcal{H}}(v,\Lambda)} \mid \big(Z_D Z_A - Z\, Z_{AD}\big).$$

Moreover, $(ZZ_A)^{-1}$ is analytic in a neighborhood of $\beta = 0$. Hence

$$\beta^{d_{\mathcal{H}}(v,\Lambda)} \mid \big(\mu_v(D) - \mu_v^A(D)\big),$$

as claimed. ◀

One-sided LDC suffices to derive the full LDC property.

Proof of Lemma 3.3. Let $S := (A \neq B) = \{u \in \Lambda : \ \nu_u(A_u \triangle B_u) > 0\}$. For every $i \in \Lambda \setminus S$, we only know $\nu_i(A_i \triangle B_i) = 0$, so A_i and B_i need not be equal as sets. Define a cylinder event C by

$$C_i = \begin{cases} A_i \cap B_i, & i \in \Lambda \setminus S, \\ X_i, & i \notin \Lambda \setminus S, \end{cases} \qquad C := \prod_{i \in V} C_i.$$

Now define cylinder events A_S, B_S supported on S by

$$(A_S)_i = \begin{cases} A_i, & i \in S, \\ X_i, & i \notin S, \end{cases} \qquad (B_S)_i = \begin{cases} B_i, & i \in S, \\ X_i, & i \notin S. \end{cases}$$

Let $\widetilde{A} := C \cap A_S$ and $\widetilde{B} := C \cap B_S$. Since $\nu_i(A_i \triangle B_i) = 0$ for every $i \in \Lambda \setminus S$, replacing A_i and B_i on these coordinates by $A_i \cap B_i$ changes the corresponding cylinder events only by a ν-null set. Hence $\nu(A \triangle \widetilde{A}) = 0$ and $\nu(B \triangle \widetilde{B}) = 0$. Consequently, $Z_A(\beta) = Z_{\widetilde{A}}(\beta)$, $Z_B(\beta) = Z_{\widetilde{B}}(\beta)$, and $\mu_v^A(D) = \mu_v^{\widetilde{A}}(D)$, $\mu_v^B(D) = \mu_v^{\widetilde{B}}(D)$ for every $D \in \mathcal{B}_v$. It therefore suffices to prove the divisibility for $\widetilde{A}, \widetilde{B}$ in place of A, B.

Since A, B are valid and the modifications are null-set, $\widetilde{A}, \widetilde{B}$ are valid, and so is C (as $\widetilde{A} \subseteq C$ and $\widetilde{A}$ is valid). Consider the conditioned product model on C; its partition functions are

$$Z_F^C(\beta) := \frac{Z_{C \cap F}(\beta)}{\nu(C)} \qquad (F \in \mathcal{B}).$$

Because C is a valid cylinder event, the reference measure $\nu(\cdot \mid C)$ is again a product probability measure on the same hypergraph. The common normalizing factor $\nu(C)$ cancels in all marginal ratios. Hence the previous one-sided lemma applies in this model. Since both A_S and B_S are supported on $S \subseteq V \setminus \{v\}$, for every $D \in \mathcal{B}_v$ we obtain

$$\beta^{d_{\mathcal{H}}(v,S)} \Bigm| \big(\mu_v^C(D) - \mu_v^{C \cap A_S}(D)\big)$$

and

$$\beta^{d_{\mathcal{H}}(v,S)} \Bigm| \big(\mu_v^C(D) - \mu_v^{C \cap B_S}(D)\big).$$

Since $C \cap A_S = \widetilde{A}$ and $C \cap B_S = \widetilde{B}$, subtracting gives

$$\beta^{d_{\mathcal{H}}(v,S)} \Bigm| \big(\mu_v^{\widetilde{A}}(D) - \mu_v^{\widetilde{B}}(D)\big).$$

Because $\mu_v^{\widetilde{A}}(D) = \mu_v^A(D)$ and $\mu_v^{\widetilde{B}}(D) = \mu_v^B(D)$, we conclude

$$\beta^{d_{\mathcal{H}}(v,S)} \Bigm| \big(\mu_v^A(D) - \mu_v^B(D)\big).$$

Since $S = (A \neq B)$, the claim follows. ◀

4 Strong spatial mixing for continuous systems

Recall Lemma 2.3. Together with the LDC property, it implies that to establish SSM it suffices to prove a uniform bound for the relevant family of functions; we will obtain this uniform bound via Montel's theorem.

We first show that, under the setting of Theorem 2.5, conditional marginals of nontrivial one-site events omit 0 and 1. Namely, let $\mathcal{H} = (V, E)$ be a hypergraph with maximum degree at most Δ and maximum edge size at most r, equipped with the product probability space $(\Omega, \mathcal{B}, \nu) := \prod_{v \in V}(X_v, \mathcal{B}_v, \nu_v)$, and with 1-Lipschitz potential functions $\{\Phi_e\}_{e \in E}$.

▶ **Lemma 4.1.** *Let $\beta \in \mathbb{D}(\Delta, r)$, $v \in V$, and A be a valid cylinder event supported on $\Lambda_A \subseteq V \setminus \{v\}$. Then for every $D \in \mathcal{B}_v$ with $0 < \nu_v(D) < 1$, $\mu_v^A(D) \notin \{0, 1\}$.*

Proof. Let $E_D := \{x \in \Omega : \ x_v \in D\}$. By the product structure, $\nu(A \cap E_D) = \nu(A)\nu_v(D) > 0$ and $\nu(A \cap \overline{E_D}) = \nu(A)\big(1 - \nu_v(D)\big) > 0$ because A is valid and $0 < \nu_v(D) < 1$. Therefore $A \cap E_D$ and $A \cap \overline{E_D}$ are valid product events. By Corollary 2.6, for this $\beta \in \mathbb{D}(\Delta, r)$, $Z_A(\beta) \neq 0, Z_{A \cap E_D}(\beta) \neq 0$ and $Z_{A \cap \overline{E_D}}(\beta) \neq 0$. Using the definition of conditional marginals, $\mu_v^A(D) = Z_{A \cap E_D}(\beta)/Z_A(\beta) \neq 0$ and $1 - \mu_v^A(D) = Z_{A \cap \overline{E_D}}(\beta)/Z_A(\beta) \neq 0$. Hence $\mu_v^A(D) \notin \{0, 1\}$. ◀

Under the setting of Theorem 2.5, conditional marginals are therefore uniformly bounded on compact subsets of the common zero-free disk.

▶ **Theorem 4.2.** *Fix integers $\Delta \geq 1$, $r \geq 2$, and a compact subset $S \subseteq \mathbb{D}(\Delta, r)$. There exists a constant $M = M(\Delta, r, S) \geq 1$ such that, for every system in the family of Theorem 1.1, every $v \in V$, every valid cylinder event A supported outside v, every $D \in \mathcal{B}_v$, and every $\beta \in S$,*

$$|\mu_v^A(D; \beta)| \leq M.$$

Proof. Let $\mathcal{F}$ consist of all functions $f(z) = \mu_v^A(D; z)$ arising from all systems and choices of v, A, D in the statement with $0 < \nu_v(D) < 1$. Each f is holomorphic on the common open domain $\mathbb{D}(\Delta, r)$ by Corollary 2.6, and omits 0 and 1 throughout that domain by Lemma 4.1. At $z = 0$, the product structure gives $f(0) = \nu_v(D) \in (0, 1)$. Applying Lemma 2.4 to this entire family yields a uniform bound on S depending only on Δ, r, S. For $\nu_v(D) \in \{0, 1\}$, the corresponding marginal is identically 0 or 1, so increasing the bound to at least 1 covers these cases as well. ◀

Combining Lemma 3.3, Theorem 4.2 and Lemma 2.3, we can establish the SSM result (Theorem 1.1).

Proof of Theorem 1.1. Fix Δ, r and $\beta_\star \in \mathbb{D}(\Delta, r)$, and set $R = (3\Delta\sqrt{r-1})^{-1}$. If $\beta_\star = 0$, the product structure gives $\mu_v^A = \mu_v^B = \nu_v$ for every permitted choice, so any $C > 0$ and $q > 1$ suffice. Otherwise, choose

$$\rho = \frac{|\beta_\star| + R}{2}, \qquad q = \frac{\rho}{|\beta_\star|} > 1.$$

Since $\rho < R$, the closed disk $\overline{\mathbb{D}}_\rho$ is contained in $\mathbb{D}(\Delta, r)$. Let $M_\rho = M(\Delta, r, \overline{\mathbb{D}}_\rho)$ be the uniform bound from Theorem 4.2, and set $C = 2M_\rho q/(q-1)$.

Now fix any permitted system, v, Λ, A, B, and put $d = d_{\mathcal{H}}(v, A \neq B)$. If $d = \infty$, the boundary events agree almost everywhere on the connected component of v; factorization over connected components gives $\mu_v^A = \mu_v^B$. For $d < \infty$ and any $D \in \mathcal{B}_v$, define $f(z) = \mu_v^A(D; z)$ and $g(z) = \mu_v^B(D; z)$. Both functions are holomorphic on $\mathbb{D}(\Delta, r)$, and Lemma 3.3 gives $z^d \mid (f - g)$. On $\partial\mathbb{D}_\rho$ they are bounded by M_ρ. Therefore Lemma 2.3 yields

$$|f(\beta_\star) - g(\beta_\star)| \leq \frac{2M_\rho}{q^{d-1}(q-1)} = Cq^{-d}.$$

Taking the supremum over $D \in \mathcal{B}_v$ proves the SSM bound. The constants were chosen using only $\beta_\star, \Delta, r$, before fixing the system or any of its boundary events. ◀

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